QCD Feynman Rules

The classical chromodynamics has a fairly simple Lagrangian

$$\mathcal{L} \;=\; \mathcal{L}_{\mathrm{Yang-Mills}} \;+ \mathcal{L}_{\mathrm{quarks}} \;=\; -\tfrac{1}{4}\sum_{\mu,\nu,a}(F^a_{\mu\nu})^2 \;+\; \sum_{i,f}\overline{\Psi}_{if}(i\not{D}+m_f)\Psi^{if} \tag{1}$$

where i denotes the color of a quark and f its flavor. $D_\mu\Psi^i = \partial_\mu\Psi^i + ig(t^a)^i{}_j\Psi^j$ where t^a are matrices representing the gauge group generators in the quark representation; in QCD the quarks belong to the fundamental $\mathbf{3}$ representation of the $SU(3)_C$ so t^a is $\frac{1}{2}\times$ Gell-Mann matrix λ^a.

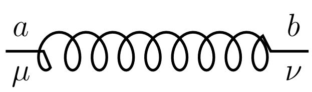
The Quantum ChromoDynamics is more complicated, even at the Lagrangian level: including the gauge-fixing and the ghost terms as well as the counterterms, we have

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4}(F^a_{\mu\nu})^2 \;-\; \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \;+\; \partial_\mu\bar{c}^a D^\mu c^a \;+\; \sum_f \overline{\Psi}_{if}(i\not{D}+m_f)\Psi^{if} \\ & -\; \frac{\delta_3}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 \;+\; g\delta_1^{(3g)} f^{abc}A^b_\mu A^c_\nu \partial_\mu A^{a\nu} \;-\; \frac{g^2\delta_1^{(4g)}}{4}(f^{abc}A^b_\mu A^c_\nu)^2 \\ & +\; \delta_2^{(\mathrm{gh})}\partial_\mu\bar{c}^a\partial^\mu c^a \;-\; g\delta_1^{(\mathrm{gh})} f^{abc}\partial_\mu\bar{c}^a A^{b\mu}c^c \\ & +\; \sum_f \overline{\Psi}_{if}\Big(i\delta_2^{(q_f)}\not{\partial} \;+\; \delta_m^{(q_f)} \;-\; g\delta_1^{(q_f)}\not{A}^a t^a\Big)\Psi^{if}.\end{aligned} \tag{2}$$

In this formula, all sums over colors (fundamental or adjoint) are implicit, as well as sums over Lorentz or Dirac indices. But the sums over quark flavors are explicit because the quark masses depend on the flavor and hence the quark-related counterterms $\delta_2^{(q_f)}$, $\delta_1^{(q_f)}$, and $\delta_m^{(q_f)}$ could also be flavor-dependent.

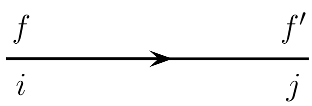
QCD Feynman rules follow from expanding the Lagrangian (2) into the free quadratic terms and the interaction terms (cubic, quartic, and all the counterterms). Thus we have:

— Gluon propagator



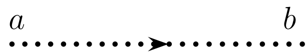
$$= \frac{-i\delta^{ab}}{k^2+i0}\left(g^{\mu\nu} \;+\; (\xi-1)\frac{k^\mu k^\nu}{k^2+i0}\right). \tag{3}$$

— Quark propagator



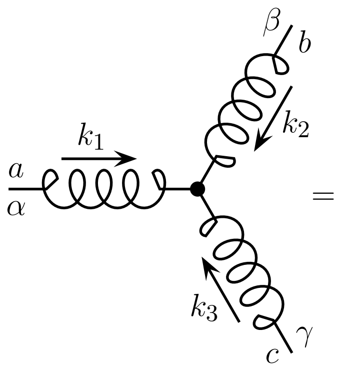
$$= \frac{i\delta^i_j\delta^f_{f'}}{\not{p}-m_f+i0}\,. \tag{4}$$

— Ghost propagator



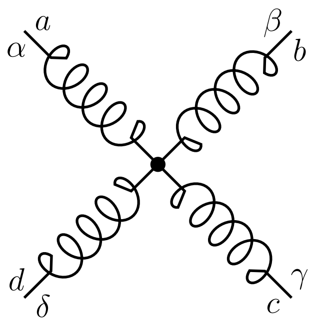
$$= \frac{i\delta^{ab}}{k^2 + i0}\,. \tag{5}$$

- Three-gluon vertex



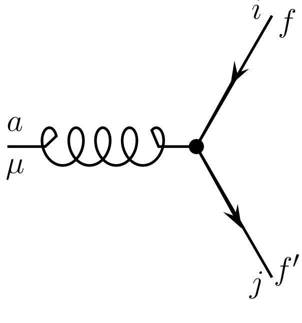
$$= -g f^{abc}\Big[g^{\alpha\beta}(k_1 - k_2)^{\gamma} \;+\; g^{\beta\gamma}(k_2 - k_3)^{\alpha} \;+\; g^{\gamma\alpha}(k_3 - k_1)^{\beta}\Big]. \tag{6}$$

- Four-gluon vertex



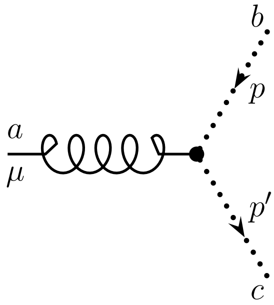
$$= -ig^2 \begin{bmatrix} f^{abe} f^{cde}(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}) \\ + f^{ace} f^{bde}(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\gamma\beta}) \\ + f^{ade} f^{bce}(g^{\alpha\beta} g^{\delta\gamma} - g^{\alpha\gamma} g^{\delta\beta}) \end{bmatrix}. \tag{7}$$

- Quark-gluon vertex



$$= -ig\gamma^{\mu} \times \delta^{f'}_{f} \times (t^a)^{j}_{\;i}\,. \tag{8}$$

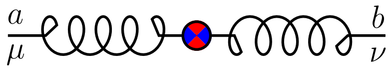
- Ghost-gluon vertex



$$= \; -g f^{abc} p'^{\mu}. \tag{9}$$

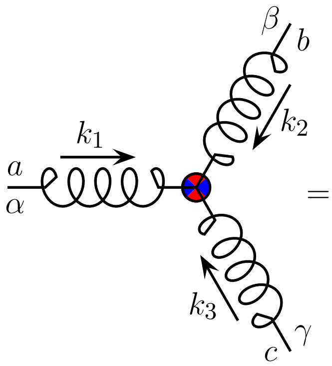
In addition, the renormalized theory has a whole bunch of the counterterm vertices:

* Two-gluon counterterm vertex



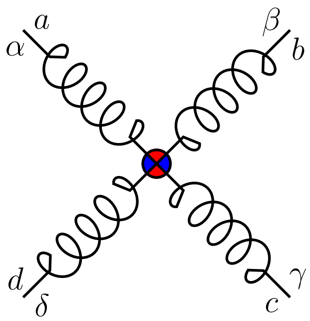
$$= \; -i\delta_3 \delta^{ab} \left(k^2 g^{\mu\nu} - k^{\mu} k^{\nu}\right). \tag{10}$$

* Three-gluon counterterm vertex



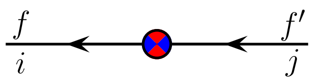
$$= \; -g\delta_1^{(3g)} \times f^{abc} \left[g^{\alpha\beta}(k_1 - k_2)^{\gamma} \; + \; g^{\beta\gamma}(k_2 - k_3)^{\alpha} \; + \; g^{\gamma\alpha}(k_3 - k_1)^{\beta} \right]. \tag{11}$$

- Four-gluon counterterm vertex



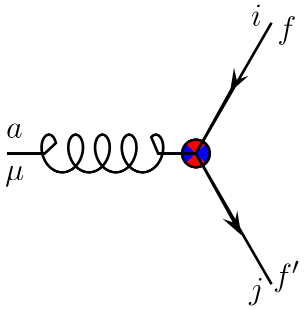
$$= \; -ig^2 \delta_1^{(4g)} \times \begin{bmatrix} f^{abe} f^{cde} (g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}) \\ + f^{ace} f^{bde} (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\gamma\beta}) \\ + f^{ade} f^{bce} (g^{\alpha\beta} g^{\delta\gamma} - g^{\alpha\gamma} g^{\delta\beta}) \end{bmatrix}. \tag{12}$$

* Two-quark counterterm vertex



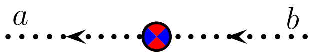
$$= \; \delta_f^{f'} \delta_i^j \times \left(i\delta_m^{(q_f)} \; - \; i\delta_2^{(q_f)} \times \not{p} \right). \tag{13}$$

* Quark-gluon counterterm vertex



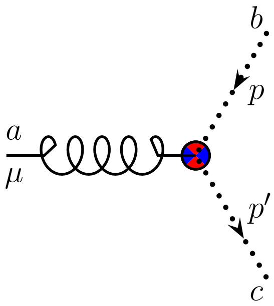
$$= -ig\delta_1^{(q_f)}\delta_f^{f'} \times \gamma^\mu \times (t^a)^j{}_i\,. \tag{14}$$

* Two ghost counterterm vertex



$$= \delta^{ab} \times i\delta_2^{(\mathrm{gh})} \times k^2. \tag{15}$$

* Ghost-gluon counterterm vertex



$$= -g\delta_1^{(\mathrm{gh})} \times f^{abc}p'^\mu. \tag{16}$$

Finally, in QCD Ward-Takahashi identities are more complicated than in QED: Instead of $\delta_1 = \delta_2$, we have

$$\left(\frac{1+\delta_1}{1+\delta_2}\right)^{(q_f)} = \left(\frac{1+\delta_1}{1+\delta_2}\right)^{(\mathrm{gh})} = \frac{1+\delta_1^{(3g)}}{1+\delta_3} = \sqrt{\frac{1+\delta_1^{(4g)}}{1+\delta_3}} = \frac{\sqrt{1+\delta_3}\times g_{\mathrm{bare}}}{g_{\mathrm{renormalized}}} \neq 1. \tag{17}$$