Feynman Rules for S Matrix Elements

All Feynman diagrams for a $k_1 + k_2 \to p_1 + \cdots + p_N$ scattering process have following features:

- Several vertices, all of valence 4.
- Several *internal* lines — connected to vertices at both ends.
- $N+2$ external lines, 2 incoming and N outgoing — only one end connected to a vertex.

⋆ No vacuum bubbles.

To evaluate a diagram:

1. Assign momenta to all lines: *fixed* momenta k^μ or p^μ for the external lines, *variable* momenta q^μ for the internal lines.

   For each internal line, choose a direction in which its momentum q^μ flows from one vertex into another; use arrows to indicate the directions of momentum flow. For the external lines, the directions are fixed: inflow for incoming particles' lines, and outflow for the outgoing particles' lines.

   $$k^\mu \longrightarrow\bullet \qquad \bullet\longrightarrow p^\mu \qquad \bullet\underset{q^\mu}{\longrightarrow}\bullet$$

2. Multiply the following factors:
   - $\dfrac{i}{q^2 - m^2 + i\epsilon}$ for each *internal* line.
   - $(-i\lambda) \times (2\pi)^4 \delta^{(4)}(\pm q_1 \pm q_2 \pm q_3 \pm q_4)$ for each vertex. Here $q_1, \ldots, q_4$ stand for momenta of lines connected to the vertex; some of them may be k's or p's if the lines are external rather than internal. The sign of each q is $+$ if the momentum flows into the vertex and $-$ if it flows out.

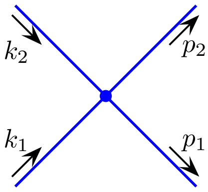
   ⋆ Combinatorical factor for the whole diagram, 1/#symmetries.

3. Integrate $\displaystyle\int \frac{d^4q}{(2\pi)^4}$ over all internal lines' momenta.

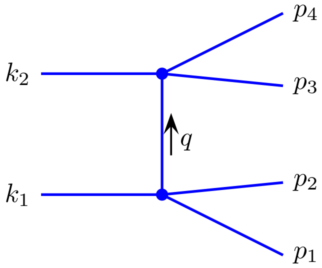
   The δ-function in vertices 'eat up' many such integrals and lead to Kirchhoff Laws for the momenta. (As if the diagram was an electric circuit and momenta were currents.)

   Ultimately, there is one non-trivial $\displaystyle\int \frac{d^4q}{(2\pi)^4}$ integral for each *closed loop* in the diagram, and one un-integrated $(2\pi)^4\delta^{(4)}(\sum \text{momenta})$ for each *connected component* of the diagram. (See examples on the next page.)

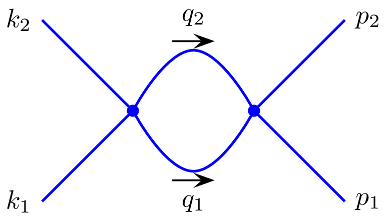
Examples of Feynman Rules



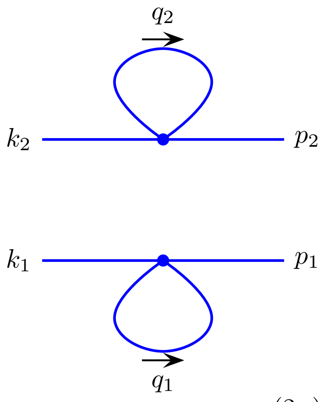
$$= (-i\lambda) \times (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \tag{1}$$



$$\begin{aligned} &= \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \times (-i\lambda)(2\pi)^4 \delta^{(4)}(k_1 - q - p_1 - p_2) \\ &\qquad \times (-i\lambda)(2\pi)^4 \delta^{(4)}(k_2 + q - p_3 - p_4) \\ &= (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2 - p_3 - p_4) \times \left. \frac{-i\lambda^2}{q^2 - m^2} \right|_{q = k_1 - p_1 - p_2 = p_3 + p_4 - k_2} \end{aligned} \tag{2}$$



$$\begin{aligned} &= \int \frac{d^4q_1}{(2\pi)^4} \frac{i}{q_1^2 - m^2 + i\epsilon} \times \int \frac{d^4q_2}{(2\pi)^4} \frac{i}{q_2^2 - m^2 + i\epsilon} \times \frac{1}{2} \\ &\qquad \times (-i\lambda)\,(2\pi)^4 \delta^{(4)}(k_1 + k_2 - q_1 - q_2) \\ &\qquad \times (-i\lambda)\,(2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2) \\ &= (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \times \frac{\lambda^2}{2} \times \int \frac{d^4q_1}{(2\pi)^4} \frac{1}{q_1^2 - m^2 + i\epsilon} \frac{1}{(k_1 + k_2 - q_1)^2 - m^2 + i\epsilon} \end{aligned} \tag{3}$$



$$\begin{aligned} &= \int \frac{d^4q_1}{(2\pi)^4} \frac{i}{q_1^2 - m^2 + i\epsilon} \times \int \frac{d^4q_2}{(2\pi)^4} \frac{i}{q_2^2 - m^2 + i\epsilon} \times \frac{1}{4} \\ &\qquad \times (-i\lambda)\,(2\pi)^4 \delta^{(4)}(k_1 + q_1 - q_1 - p_1) \\ &\qquad \times (-i\lambda)\,(2\pi)^4 \delta^{(4)}(k_2 + q_2 - q_2 - p_2) \\ &= (2\pi)^4 \delta^{(4)}(k_1 - p_1) \times (2\pi)^4 \delta^{(4)}(k_2 - p_2) \times \frac{\lambda^2}{4} \times \left[\int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m^2 + i\epsilon} \right]^2 \end{aligned} \tag{4}$$