

Feynman Rules for S Matrix Elements

All Feynman diagrams for a $k_1 + k_2 \rightarrow p_1 + \dots + p_N$ scattering process have following features:

- Several vertices, all of valence 4.
- Several *internal* lines — connected to vertices at both ends.
- $N + 2$ external lines, 2 incoming and N outgoing — only one end connected to a vertex.
- ★ No vacuum bubbles.

To evaluate a diagram:

1. Assign momenta to all lines: *fixed* momenta k^μ or p^μ for the external lines, *variable* momenta q^μ for the internal lines.

For each internal line, choose a direction in which its momentum q^μ flows from one vertex into another; use arrows to indicate the directions of momentum flow. For the external lines, the directions are fixed: inflow for incoming particles' lines, and outflow for the outgoing particles' lines.

$$k^\mu \longrightarrow \bullet \qquad \bullet \longrightarrow p^\mu \qquad \bullet \xrightarrow{q^\mu} \bullet$$

2. Multiply the following factors:

- $\frac{i}{q^2 - m^2 + i\epsilon}$ for each *internal* line.
- $(-i\lambda) \times (2\pi)^4 \delta^{(4)}(\pm q_1 \pm q_2 \pm q_3 \pm q_4)$ for each vertex. Here q_1, \dots, q_4 stand for momenta of lines connected to the vertex; some of them may be k 's or p 's if the lines are external rather than internal. The sign of each q is $+$ if the momentum flows into the vertex and $-$ if it flows out.

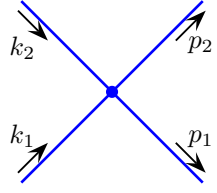
★ Combinatorial factor for the whole diagram, $1/\#\text{symmetries}$.

3. Integrate $\int \frac{d^4q}{(2\pi)^4}$ over all internal lines' momenta.

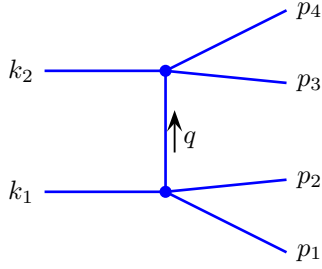
The δ -function in vertices 'eat up' many such integrals and lead to [Kirchhoff Laws](#) for the momenta. (As if the diagram was an electric circuit and momenta were currents.)

Ultimately, there is one non-trivial $\int \frac{d^4q}{(2\pi)^4}$ integral for each *closed loop* in the diagram, and one un-integrated $(2\pi)^4 \delta^{(4)}(\sum \text{momenta})$ for each *connected component* of the diagram. (See examples on the next page.)

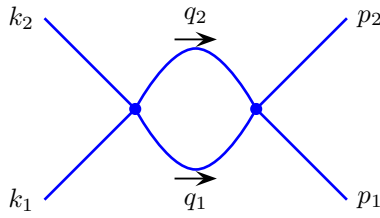
Examples of Feynman Rules



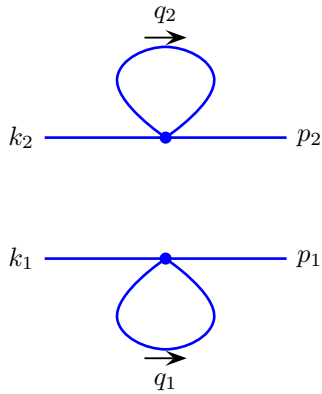
$$= (-i\lambda) \times (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \quad (1)$$



$$\begin{aligned} &= \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \times (-i\lambda)(2\pi)^4 \delta^{(4)}(k_1 - q - p_1 - p_2) \\ &\quad \times (-i\lambda)(2\pi)^4 \delta^{(4)}(k_2 + q - p_3 - p_4) \\ &= (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2 - p_3 - p_4) \times \frac{-i\lambda^2}{q^2 - m^2} \Big|_{q=k_1-p_1-p_2=p_3+p_4-k_2} \end{aligned} \quad (2)$$



$$\begin{aligned} &= \int \frac{d^4 q_1}{(2\pi)^4} \frac{i}{q_1^2 - m^2 + i\epsilon} \times \int \frac{d^4 q_2}{(2\pi)^4} \frac{i}{q_2^2 - m^2 + i\epsilon} \times \frac{1}{2} \\ &\quad \times (-i\lambda)(2\pi)^4 \delta^{(4)}(k_1 + k_2 - q_1 - q_2) \\ &\quad \times (-i\lambda)(2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2) \\ &= (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \times \frac{\lambda^2}{2} \times \int \frac{d^4 q_1}{(2\pi)^4} \frac{1}{q_1^2 - m^2 + i\epsilon} \frac{1}{(k_1 + k_2 - q_1)^2 - m^2 + i\epsilon} \end{aligned} \quad (3)$$



$$\begin{aligned} &= \int \frac{d^4 q_1}{(2\pi)^4} \frac{i}{q_1^2 - m^2 + i\epsilon} \times \int \frac{d^4 q_2}{(2\pi)^4} \frac{i}{q_2^2 - m^2 + i\epsilon} \times \frac{1}{4} \\ &\quad \times (-i\lambda)(2\pi)^4 \delta^{(4)}(k_1 + q_1 - q_1 - p_1) \\ &\quad \times (-i\lambda)(2\pi)^4 \delta^{(4)}(k_2 + q_2 - q_2 - p_2) \\ &= (2\pi)^4 \delta^{(4)}(k_1 - p_1) \times (2\pi)^4 \delta^{(4)}(k_2 - p_2) \times \frac{\lambda^2}{4} \times \left[\int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 - m^2 + i\epsilon} \right]^2 \end{aligned} \quad (4)$$