Problem $\mathbf{1}$(a):

First, let's verify eq. (3) for a state $|\gamma_1,\ldots,\gamma_N\rangle$, with wave-function

$$\psi(\mathbf{x}_1,\ldots,\mathbf{x}_N) \;=\; \frac{1}{T\sqrt{D}}\times\phi_{(\gamma_1}(\mathbf{x}_1)\cdots\phi_{\gamma_N)}(\mathbf{x}_N) \tag{S.1}$$

where () surrounding the indices $(\gamma_1\cdots\gamma_N)$ denote total symmetrization, *i.e.* summing over all $N!$ permutations, T is the number of trivial permutations (of indices which happen to coincide), and D is the number of distinct permutations (of indices which do not coincide). For this state,

$$\hat{a}_\alpha\,|\gamma_1,\ldots,\gamma_N\rangle \;=\; \sqrt{n_\alpha+1}\,|\gamma_1,\ldots,\gamma_N,\alpha\rangle\,, \tag{S.2}$$

which has wave-function

$$\psi'(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{x}_{N+1}) \;=\; \frac{\sqrt{n_\alpha+1}}{T'\sqrt{D'}}\times\phi_{(\gamma_1}(\mathbf{x}_1)\cdots\phi_{\gamma_N}(\mathbf{x}_N)\phi_{\alpha)}(\mathbf{x}_{N+1}). \tag{S.3}$$

Here we sum over $N+1$ permutations of indices $(\gamma_1\cdots\gamma_N\alpha)$, which may be done in $N+1$ blocks of $N!$, namely first permute the γ's among themselves, and then put α anywhere in that list,

$$\phi_{(\gamma_1}(\mathbf{x}_1)\cdots\phi_{\gamma_N}(\mathbf{x}_N)\phi_{\alpha)}(\mathbf{x}_{N+1}) \;=\; \sum_{i=1}^{N+1}\phi_\alpha(\mathbf{x}_i)\times\phi_{(\gamma_1}(\mathbf{x}_1)\cdots\cancel{\phi(\mathbf{x}_i)}\cdots\phi_{\gamma_N)}(\mathbf{x}_{N+1}). \tag{S.4}$$

But the symmetrization over γ's here is exactly as in eq. (S.1), except for the relevant coordinates being $(\mathbf{x}_1,\ldots,\cancel{\mathbf{x}}_i,\ldots,\mathbf{x}_{N+1})$ instead of $(\mathbf{x}_1,\ldots,\mathbf{x}_N)$. Therefore,

$$\psi'(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{x}_{N+1}) \;=\; \frac{\sqrt{n_\alpha+1}}{T'\sqrt{D'}}\times T\sqrt{D}\times\sum_{i=1}^{N+1}\phi_\alpha(\mathbf{x}_i)\times\psi(\mathbf{x}_1,\ldots,\cancel{\mathbf{x}}_i,\ldots,\mathbf{x}_N), \tag{S.5}$$

exactly as in eq. (3), except maybe the overall coefficient. To check this coefficient, we use eqs. (2). Given occupation numbers n_β of the original state $|\gamma_1,\ldots,\gamma_N\rangle$, the new state

$|\gamma_1,\ldots,\gamma_N,\alpha\rangle$ has $n'_\beta = n_\beta + \delta_{\alpha\beta}$, hence

$$\begin{aligned}
\frac{T'}{T} \;&=\; \prod_\beta n'_\beta! \Big/ \prod_\beta n_\beta! \;=\; \frac{(n_\alpha+1)!}{n_\alpha!} \;=\; n_\alpha + 1, \\
\frac{T'\sqrt{D'}}{T\sqrt{D}} \;&=\; \sqrt{\frac{T'\times(N+1)!}{T\times N!}} \;=\; \sqrt{(n_\alpha+1)(N+1)}\,, \\
\frac{\sqrt{n_\alpha+1}}{T'\sqrt{D'}}\times T\sqrt{D} \;&=\; \frac{1}{\sqrt{N+1}}\,.
\end{aligned} \tag{S.6}$$

Thus, the coefficient in eq. (S.5) is also exactly as in eq. (3).

At this point, we have proved eq. (3) for states $|N,\Psi\rangle$ that happen to be $|\gamma_1,\ldots,\gamma_N\rangle$ for some $\gamma_1,\ldots,\gamma_N$. To prove it for all N-boson states $|N,\psi\rangle$ we now use *linearity:* the operator $\hat{a}^\dagger_\alpha$ is linear, and eq. (3) is manifestly linear with respect to ψ and ψ', so if it holds for any set of states, it also holds for all their linear combinations. But states $|\gamma_1,\ldots,\gamma_N\rangle$ make up a complete basis of the N-boson Hilbert space, so any $|N,\psi\rangle$ is a linear combination of such states. Therefore, eq. (3) must hold for any N-boson wave function $\psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)$. $\mathcal{Q.E.D.}$

Problem **1**(b):

The operator $\hat{a}_\alpha$ is the hermitian conjugate of the operator $\hat{a}^\dagger_\alpha$, so for any two states $|N,\psi\rangle$ and $\langle N-1,\widetilde{\psi}|$,

$$\langle N-1,\widetilde{\psi}|\,\hat{a}_\alpha\,|N,\psi\rangle \;=\; \langle N,\psi|\,\hat{a}_\alpha\,|N-1,\widetilde{\psi}\rangle^*\,. \tag{S.7}$$

In wave-function terms, this means

$$\begin{aligned}
&\int\! d^3\mathbf{x}_1 \cdots\! \int\! d^3\mathbf{x}_{N-1}\, \widetilde{\psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \times \psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;= \\
&\qquad = \int\! d^3\mathbf{x}_1 \cdots\! \int\! d^3\mathbf{x}_N\, \psi(\mathbf{x}_1,\ldots,\mathbf{x}_N) \times \left[\widetilde{\psi}'(\mathbf{x}_1,\ldots,\mathbf{x}_N)\right]^*
\end{aligned} \tag{S.8}$$

where $\widetilde{\psi}'(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ is the wave function of the state $\left|N,\widetilde{\psi}'\right\rangle = \hat{a}^\dagger_\alpha\,|N-1,\widetilde{\psi}\rangle$. Applying eq. (3)

of part (a) to this wave-function, we obtain

$$\begin{aligned}
&\int\! d^3\mathbf{x}_1 \cdots \int\! d^3\mathbf{x}_{N-1}\, \widetilde{\psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \times \psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;= \\
&= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int\! d^3\mathbf{x}_1 \cdots \int\! d^3\mathbf{x}_N\, \psi(\mathbf{x}_1,\ldots,\mathbf{x}_N) \times \phi^*_\alpha(\mathbf{x}_i) \times \widetilde{\psi}^*(\mathbf{x}_1,\ldots,\not{\mathbf{x}}_i,\ldots,\mathbf{x}_N) \\
&\text{by permutational symmetry} \\
&= \frac{N}{\sqrt{N}} \int\! d^3\mathbf{x}_1 \cdots \int\! d^3\mathbf{x}_N\, \psi(\mathbf{x}_1,\ldots,\mathbf{x}_N) \times \phi^*_\alpha(\mathbf{x}_N) \times \widetilde{\psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \\
&= \int\! d^3\mathbf{x}_1 \cdots \int\! d^3\mathbf{x}_{N-1}\, \widetilde{\psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \times \sqrt{N}\int\! d^3\mathbf{x}_N \phi^*_\alpha(\mathbf{x}_N) \times \psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N).
\end{aligned} \tag{S.9}$$

This formula holds true for any totally symmetric wave-function $\widetilde{\psi}(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})$, and this is possible only when

$$\psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;=\; \sqrt{N}\int\! d^3\mathbf{x}_N \phi^*_\alpha(\mathbf{x}_N) \times \psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N), \tag{4}$$

or rather when the totally symmetric part of the left hand side here equals to the totally symmetric part of the right hand side. But for bosonic wave functions ψ and $\widetilde{\psi}$ both sides must be already totally symmetric in $(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})$ as they are, so eq. (4) must apply exactly as written. $\mathcal{Q.E.D.}$

Problem **1**(c):

Let $A_{\alpha\beta} = \langle\alpha|\,\hat{A}_1\,|\beta\rangle$. Since states $|\alpha\rangle$ make a complete basis of the 1-particle Hilbert space, for any 1-particle states $\langle\widetilde{\psi}|$ and $|\psi\rangle$

$$\langle\widetilde{\psi}|\,\hat{A}_1\,|\psi\rangle \;=\; \sum_{\alpha,\beta} A_{\alpha\beta}\,\langle\widetilde{\psi}|\alpha\rangle\,\langle\beta|\psi\rangle \;=\; \sum_{\alpha,\beta} A_{\alpha\beta} \times \int\! d^3\tilde{\mathbf{x}}\,\widetilde{\psi}^*(\tilde{\mathbf{x}})\phi_\alpha(\tilde{\mathbf{x}}) \times \int\! d^3\mathbf{x}\,\phi^*_\beta(\mathbf{x})\psi(\mathbf{x}). \tag{S.10}$$

This is undergraduate-level QM.

In the N–particle Hilbert space we have a similar formula for the matrix elements of the $\hat{A}_1$ acting on particle $\#i$, the only modification being integrals over the coordinates of the other particles,

$$
\begin{aligned}
\langle N,\widetilde{\psi}|\,\hat{A}_1(i^{\rm th})\,|N,\psi\rangle &= \\
&= \int\!\cdots\!\int\! d^3\mathbf{x}_1\cdots \cancel{d^3\mathbf{x}_i}\cdots d^3\mathbf{x}_N \sum_{\alpha,\beta} A_{\alpha\beta}\times\left(\int\! d^3\tilde{\mathbf{x}}_i\,\widetilde{\psi}^*(\mathbf{x}_1,\ldots,\tilde{\mathbf{x}}_i,\ldots,\mathbf{x}_N)\phi_\alpha(\tilde{\mathbf{x}}_i)\right)\\
&\qquad\qquad\times\left(\int\! d^3\mathbf{x}_i\,\phi^*_\beta(\mathbf{x}_i)\psi(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_N)\right)\\
&= \sum_{\alpha,\beta} A_{\alpha\beta}\times\int\!\cdots\!\int\! d^3\mathbf{x}_1\cdots d^3\mathbf{x}_N\,d^3\tilde{\mathbf{x}}_i\,\widetilde{\psi}^*(\mathbf{x}_1,\ldots,\tilde{\mathbf{x}}_i,\ldots,\mathbf{x}_N)\times\phi_\alpha(\tilde{\mathbf{x}}_i)\\
&\qquad\qquad\times\phi^*_\beta(\mathbf{x}_i)\times\psi(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_N).
\end{aligned}
\tag{S.11}
$$

For symmetric wave-functions of identical bosons, we get the same matrix element regardless of which particle $\#i$ we are acting on with the operator $\hat{A}_1$, hence for the *net* A operator (5),

$$
\begin{aligned}
\langle N,\widetilde{\psi}|\,\hat{A}^{(1)}_{\rm net}\,|N,\psi\rangle \;=\; N\times\sum_{\alpha,\beta}A_{\alpha\beta}\times\int\!\cdots\!\int\! d^3\mathbf{x}_1\cdots d^3\mathbf{x}_{N-1}\,d^3\mathbf{x}_N d^3\tilde{\mathbf{x}}_N\\
\widetilde{\psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\tilde{\mathbf{x}}_N)\times\phi_\alpha(\tilde{\mathbf{x}}_N)\\
\times\,\phi^*_\beta(\mathbf{x}_N)\times\psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N).
\end{aligned}
\tag{S.12}
$$

Now consider matrix elements of the Fock-space operator (6). According to eq. (4) of part (b), state $|N-1,\psi''\rangle=\hat{a}_\beta\,|N,\psi\rangle$ has wave-function

$$
\psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;=\; \sqrt{N}\int\! d^3\mathbf{x}_N\,\phi^*_\beta(\mathbf{x}_N)\times\psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N). \tag{S.13}
$$

Likewise, state $\left|N-1,\widetilde{\psi}''\right\rangle=\hat{a}_\alpha\,|N,\widetilde{\psi}\rangle$ has wave-function

$$
\widetilde{\psi}''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;=\; \sqrt{N}\int\! d^3\tilde{\mathbf{x}}_N\,\phi^*_\alpha(\tilde{\mathbf{x}}_N)\times\widetilde{\psi}(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\tilde{\mathbf{x}}_N), \tag{S.14}
$$

and consequently

$$
\begin{aligned}
\langle N, \widetilde{\psi}|\, \hat{a}^\dagger_\alpha \hat{a}_\beta \,|N, \psi\rangle \;&=\; \big\langle N-1, \widetilde{\psi}''\big|\,\big|N-1, \psi''\big\rangle \\
&= \int\!\cdots\!\int d^3\mathbf{x}_1 \cdots \mathbf{x}_{N-1}\, \widetilde{\psi}''^{*}(\mathbf{x}_1, \ldots, \mathbf{x}_{N-1}) \times \psi''(\mathbf{x}_1, \ldots, \mathbf{x}_{N-1}) \\
&= \int\!\cdots\!\int d^3\mathbf{x}_1 \cdots \mathbf{x}_{N-1}\, \sqrt{N} \int d^3\tilde{\mathbf{x}}_N\, \phi_\alpha(\tilde{\mathbf{x}}_N) \times \widetilde{\psi}^{*}(\mathbf{x}_1, \ldots, \mathbf{x}_{N-1}, \tilde{\mathbf{x}}_N) \times \\
&\qquad\qquad \times \sqrt{N} \int d^3\mathbf{x}_N\, \phi^*_\beta(\mathbf{x}_N) \times \psi(\mathbf{x}_1, \ldots, \mathbf{x}_{N-1}, \mathbf{x}_N).
\end{aligned}
\tag{S.15}
$$

Comparing this formula to the integrals in eq. (S.12), we see that

$$
\langle N, \widetilde{\psi}|\, \hat{A}^{(1)}_{\rm net} \,|N, \psi\rangle \;=\; \sum_{\alpha,\beta} A_{\alpha\beta} \times \langle N, \widetilde{\psi}|\, \hat{a}^\dagger_\alpha \hat{a}_\beta \,|N, \psi\rangle \;=\; \langle N, \widetilde{\psi}|\, \hat{A}^{(2)}_{\rm net} \,|N, \psi\rangle\,. \tag{S.16}
$$

$\mathcal{Q.E.D.}$

Problem **1**(d):

This works similarly to part (c), except for more integrals ☹. Let

$$
B_{\alpha\beta,\gamma\delta} \;=\; \big(\langle\alpha| \otimes \langle\beta|\big)\hat{B}_2\big(|\gamma\rangle \otimes |\delta\rangle\big) \tag{S.17}
$$

be matrix elements of a two-body operator $\hat{B}_2$ between *un-symmetrized* two-particle states. Then for generic two-particle states $\langle\widetilde{\psi}|$ and $|\psi\rangle$ — symmetric or not — we have

$$
\begin{aligned}
\langle\widetilde{\psi}|\, \hat{B}_2 \,|\psi\rangle \;&=\; \sum_{\alpha,\beta,\gamma,\delta} B_{\alpha\beta,\gamma\delta} \times \langle\widetilde{\psi}|\, \big(|\alpha\rangle \otimes |\beta\rangle\big) \times \big(\langle\gamma| \otimes \langle\delta|\big)\, |\psi\rangle \\
&= \sum_{\alpha,\beta,\gamma,\delta} B_{\alpha\beta,\gamma\delta} \times \iint d^3\tilde{\mathbf{x}}_1\, d^3\tilde{\mathbf{x}}_2\, \widetilde{\psi}^*(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2)\phi_\alpha(\tilde{\mathbf{x}}_1)\phi_\beta(\tilde{\mathbf{x}}_2) \\
&\qquad \times \iint d^3\mathbf{x}_1\, d^3\mathbf{x}_2\, \phi^*_\gamma(\mathbf{x}_1)\phi^*_\delta(\mathbf{x}_2)\psi(\mathbf{x}_1, \mathbf{x}_2).
\end{aligned}
\tag{S.18}
$$

Similarly, in the Hilbert space of $N > 2$ particles — identical bosons or not — operators $\hat{B}_2$

acting on particles $\#i$ and $\#j$ has matrix elements

$$\begin{aligned}
\langle N,\widetilde{\psi}|\,\hat{B}_2(i^{\rm th},j^{\rm th})\,|N,\psi\rangle \;&= \\
= \sum_{\alpha,\beta,\gamma,\delta} B_{\alpha\beta,\gamma\delta}\times \int\!\cdots\!\int & d^3\mathbf{x}_1\cdots \cancel{d^3\mathbf{x}_i}\cdots \cancel{d^3\mathbf{x}_j}\cdots d^3\mathbf{x}_N \\
&\iint d^3\tilde{\mathbf{x}}_i\, d^3\tilde{\mathbf{x}}_j\, \widetilde{\psi}^*(\mathbf{x}_1,\ldots,\tilde{\mathbf{x}}_i,\ldots,\tilde{\mathbf{x}}_j,\ldots,\mathbf{x}_N)\phi_\alpha(\tilde{\mathbf{x}}_i)\phi_\beta(\tilde{\mathbf{x}}_j) \\
&\times\iint d^3\mathbf{x}_i\, d^3\mathbf{x}_j\, \phi^*_\gamma(\mathbf{x}_i)\phi^*_\delta(\mathbf{x}_j)\psi(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_j,\ldots,\mathbf{x}_N)
\end{aligned} \tag{S.19}$$

For identical bosons — and hence totally symmetric wave-functions ψ and $\widetilde{\psi}$ — such matrix elements do not depend on the choice of particles on which $\hat{B}_2$ acts, so we may just as well let $i=N-1$ and $j=N$. Consequently, the *net* $\hat{B}$ operator (7) has matrix elements

$$\begin{aligned}
\langle N,\widetilde{\psi}|\,\hat{B}^{(1)}_{\rm net}\,|N,\psi\rangle \;&=\; \frac{N(N-1)}{2}\times\langle N,\widetilde{\psi}|\,\hat{B}_2(N-1,N)\,|N,\psi\rangle \\
&=\; \frac{N(N-1)}{2}\sum_{\alpha,\beta,\gamma,\delta} B_{\alpha\beta,\gamma\delta}\times I_{\alpha\beta,\gamma\delta}
\end{aligned} \tag{S.20}$$

where

$$\begin{aligned}
I_{\alpha\beta,\gamma\delta} \;=\; \int\!\cdots\!\int d^3\mathbf{x}_1\cdots d^3\mathbf{x}_{N-2} \iint & d^3\tilde{\mathbf{x}}_{N-1}\, d^3\tilde{\mathbf{x}}_N\, \widetilde{\psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2},\tilde{\mathbf{x}}_{N-1},\tilde{\mathbf{x}}_N)\phi_\alpha(\tilde{\mathbf{x}}_{N-1})\phi_\beta(\tilde{\mathbf{x}}_N) \\
&\times\iint d^3\mathbf{x}_{N-1}\, d^3\mathbf{x}_N\, \phi^*_\gamma(\mathbf{x}_{N-1})\phi^*_\delta(\mathbf{x}_N)\psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2},\mathbf{x}_{N-1},\mathbf{x}_N)
\end{aligned}$$

Now let's compare this to the Fock space formalism. Applying eq. (4) of part (b) *twice*, we find that the $(N-2)$–particle state

$$\left|N-2,\psi'''\right\rangle \;=\; \hat{a}_\delta\hat{a}_\gamma\,|N,\psi\rangle \tag{S.21}$$

has wave function

$$\begin{aligned}
\psi'''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2}) \;=\; \sqrt{N(N-1)}\iint d^3\mathbf{x}_{N-1}\, d^3\mathbf{x}_N\, \phi^*_\gamma(\mathbf{x}_{N-1})\phi^*_\delta(\mathbf{x}_N) & \\
\times\,\psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2},\mathbf{x}_{N-1},\mathbf{x}_N). &
\end{aligned} \tag{S.22}$$

Likewise, state

$$\left|N-2,\widetilde{\psi}'''\right\rangle \;=\; \hat{a}_{\beta}\hat{a}_{\alpha}\left|N,\widetilde{\psi}\right\rangle \tag{S.23}$$

has wave function

$$\begin{aligned}\widetilde{\psi}'''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2}) \;=\; \sqrt{N(N-1)}\iint d^3\mathbf{x}_{N-1}\,d^3\mathbf{x}_N\,\phi^*_{\beta}(\tilde{\mathbf{x}}_{N-1})\phi^*_{\alpha}(\tilde{\mathbf{x}}_N)\\ \times\,\widetilde{\psi}(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2},\tilde{\mathbf{x}}_{N-1},\tilde{\mathbf{x}}_N).\end{aligned} \tag{S.24}$$

Taking Dirac product of these two states, we obtain

$$\begin{aligned}\langle N,\widetilde{\psi}|\,\hat{a}^{\dagger}_{\alpha}\hat{a}^{\dagger}_{\beta}\hat{a}_{\delta}\hat{a}_{\gamma}\,|N,\psi\rangle \;&=\; \left\langle N-1,\widetilde{\psi}'''\right|\left|N-2,\psi'''\right\rangle\\ &=\int\!\cdots\!\int d^3\mathbf{x}_1\cdots d^3\mathbf{x}_{N-2}\,\widetilde{\psi}'''^{*}(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2})\times\psi'''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2})\\ &=\; N(N-1)\times I_{\alpha\beta,\gamma\delta}\end{aligned} \tag{S.25}$$

where $I_{\alpha\beta,\gamma\delta}$ is exactly the same mega-integral as in eq. (S.20). Therefore,

$$\langle N,\widetilde{\psi}|\,\hat{B}^{(1)}_{\rm net}\,|N,\psi\rangle \;=\; \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta}B_{\alpha\beta,\gamma\delta}\times\langle N,\widetilde{\psi}|\,\hat{a}^{\dagger}_{\alpha}\hat{a}^{\dagger}_{\beta}\hat{a}_{\delta}\hat{a}_{\gamma}\,|N,\psi\rangle \;=\; \langle N,\widetilde{\psi}|\,\hat{B}^{(2)}_{\rm net}\,|N,\psi\rangle \tag{S.26}$$

where the second equality follows from eq. (8). $\mathcal{Q.E.D.}$

Problem **2**(a):

This is a simple exercise of the Leibniz rule for commutators:

$$\begin{aligned}[\hat{a}^{\dagger}_{\alpha}\hat{a}_{\beta},\hat{a}^{\dagger}_{\gamma}] \;&=\; [\hat{a}^{\dagger}_{\alpha},\hat{a}^{\dagger}_{\gamma}]\hat{a}_{\beta} \,+\, \hat{a}^{\dagger}_{\alpha}[\hat{a}_{\beta},\hat{a}^{\dagger}_{\gamma}] \;=\; 0 \,+\, \hat{a}^{\dagger}_{\alpha}\delta_{\beta,\gamma} \;=\; \delta_{\beta,\gamma}\hat{a}^{\dagger}_{\alpha}\,,\\ [\hat{a}^{\dagger}_{\alpha}\hat{a}_{\beta},\hat{a}_{\delta}] \;&=\; [\hat{a}^{\dagger}_{\alpha},\hat{a}_{\delta}]\hat{a}_{\beta} \,+\, \hat{a}^{\dagger}_{\alpha}[\hat{a}_{\beta},\hat{a}_{\delta}] \;=\; -\delta_{\alpha,\delta}\hat{a}_{\beta} \,+\, 0 \;=\; -\delta_{\alpha,\delta}\hat{a}_{\beta}\,,\\ [\hat{a}^{\dagger}_{\alpha}\hat{a}_{\beta},\hat{a}^{\dagger}_{\gamma}\hat{a}_{\delta}] \;&=\; [\hat{a}^{\dagger}_{\alpha}\hat{a}_{\beta},\hat{a}^{\dagger}_{\gamma}]\hat{a}_{\delta} \,+\, \hat{a}^{\dagger}_{\gamma}[\hat{a}^{\dagger}_{\alpha}\hat{a}_{\beta},\hat{a}_{\gamma}\hat{a}_{\delta}] \;=\; \delta_{\beta,\gamma}\hat{a}^{\dagger}_{\alpha}\hat{a}_{\delta} \,-\, \delta_{\alpha,\delta}\hat{a}^{\dagger}_{\gamma}\hat{a}_{\beta}\,.\end{aligned} \tag{S.27}$$

Problem **2**(b):

Given

$$\hat{A}^{(2)}_{\rm tot} = \sum_{\alpha,\beta} \langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\hat{a}^\dagger_\alpha \hat{a}_\beta \tag{S.28}$$

and

$$\hat{B}^{(2)}_{\rm tot} = \sum_{\gamma,\delta} \langle\gamma|\,\hat{B}_1\,|\delta\rangle\;\hat{a}^\dagger_\gamma \hat{a}_\delta\,, \tag{S.29}$$

we immediately have

$$\begin{aligned}
\left[\hat{A}^{(2)}_{\rm tot}, \hat{B}^{(2)}_{\rm tot}\right] &= \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\langle\gamma|\,\hat{B}_1\,|\delta\rangle\;[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta] \\
&\langle\!\langle \text{using (S.27)} \rangle\!\rangle \\
&= \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\langle\gamma|\,\hat{B}_1\,|\delta\rangle\;\Big(\delta_{\beta,\gamma}\hat{a}^\dagger_\alpha \hat{a}_\delta \;-\; \delta_{\alpha,\delta}\hat{a}^\dagger_\gamma \hat{a}_\beta\Big) \\
&= \sum_{\alpha,\delta} \hat{a}^\dagger_\alpha \hat{a}_\delta \times \sum_{\beta=\gamma} \langle\alpha|\,\hat{A}_1\,|\gamma\rangle\,\langle\gamma|\,\hat{B}_1\,|\delta\rangle \;-\sum_{\beta,\gamma} \hat{a}^\dagger_\gamma \hat{a}_\beta \times \sum_{\alpha=\delta} \langle\gamma|\,\hat{B}_1\,|\alpha\rangle\,\langle\alpha|\,\hat{A}_1\,|\beta\rangle \\
&= \sum_{\alpha,\delta} \hat{a}^\dagger_\alpha \hat{a}_\delta\;\langle\alpha|\,\hat{A}_1\hat{B}_1\,|\delta\rangle \;-\sum_{\beta,\gamma} \hat{a}^\dagger_\gamma \hat{a}_\beta\;\langle\gamma|\,\hat{B}_1\hat{A}_1\,|\beta\rangle \\
&\langle\!\langle \text{renaming summation indices} \rangle\!\rangle \\
&= \sum_{\alpha,\beta} \hat{a}^\dagger_\alpha \hat{a}_\beta \times \Big(\langle\alpha|\,\hat{A}_1\hat{B}_1\,|\beta\rangle \;-\; \langle\alpha|\,\hat{B}_1\hat{A}_1\,|\beta\rangle\Big) \\
&= \sum_{\alpha,\beta} \hat{a}^\dagger_\alpha \hat{a}_\beta \times \langle\alpha|\Big([\hat{A}_1, \hat{B}_1] = \hat{C}_1\Big)|\beta\rangle \;\equiv\; \hat{C}^{(2)}_{\rm tot}.
\end{aligned} \tag{S.30}$$

Problem **2**(c):

Again, we apply the Leibniz rule:

$$\begin{aligned}
[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta] =\; & [\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha]\hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta + \hat{a}^\dagger_\alpha[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\beta]\hat{a}_\gamma \hat{a}_\delta \\
& + \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}_\gamma]\hat{a}_\delta + \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}_\delta] \\
=\; & \delta_{\nu\alpha}\hat{a}^\dagger_\mu \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta + \delta_{\nu\beta}\hat{a}^\dagger_\alpha \hat{a}^\dagger_\mu \hat{a}_\gamma \hat{a}_\delta - \delta_{\mu\gamma}\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\nu \hat{a}_\delta - \delta_{\mu\delta}\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\nu\,.
\end{aligned} \tag{S.31}$$

Problem **2**(d):

In the Fock space,

$$\hat{A}^{(2)}_{\rm tot} \;=\; \sum_{\mu nu} \langle\mu|\,\hat{A}_1\,|\nu\rangle\ \hat{a}^\dagger_\mu \hat{a}_\nu \tag{2}$$

and

$$\hat{B}^{(2)}_{\rm tot} \;=\; \tfrac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle\ \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta\,, \tag{5}$$

where $\langle\alpha\otimes\beta|$ is a short-hand for the un-symmetrized two-particle wave function $(\langle\alpha|\otimes\langle\beta|)$ and likewise $|\gamma\otimes\delta\rangle = (|\gamma\rangle\otimes|\delta\rangle)$. Therefore,

$$\begin{aligned}
[\hat{A}^{(2)}_{\rm tot}, \hat{B}^{(2)}_{\rm tot}] \;&=\; \tfrac{1}{2} \sum_{\mu,\nu,\alpha,\beta,\gamma,\delta} \langle\mu|\,\hat{A}_1\,|\nu\rangle\ \langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle\ [\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta] \\
&\langle\!\langle \text{using eq. (S.31)} \rangle\!\rangle \\
&=\; \tfrac{1}{2} \sum_{\mu,\beta,\gamma,\delta} \hat{a}^\dagger_\mu \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta \times \sum_\nu \langle\mu|\,\hat{A}_1\,|\nu\rangle\,\langle\nu\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle \\
&\quad+\; \tfrac{1}{2} \sum_{\alpha,\mu,\gamma,\delta} \hat{a}^\dagger_\alpha \hat{a}^\dagger_\mu \hat{a}_\gamma \hat{a}_\delta \times \sum_\nu \langle\mu|\,\hat{A}_1\,|\nu\rangle\,\langle\alpha\otimes\nu|\,\hat{B}_2\,|\gamma\otimes\delta\rangle \\
&\quad-\; \tfrac{1}{2} \sum_{\alpha,\beta,\nu,\delta} \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\nu \hat{a}_\delta \times \sum_\mu \langle\alpha\otimes\beta|\,\hat{B}_2\,|\mu\otimes\delta\rangle\,\langle\mu|\,\hat{A}_1\,|\nu\rangle \\
&\quad-\; \tfrac{1}{2} \sum_{\alpha,\beta,\gamma,\nu} \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\nu \times \sum_\mu \langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\mu\rangle\,\langle\mu|\,\hat{A}_1\,|\nu\rangle \\
&\langle\!\langle \text{renaming summation indices} \rangle\!\rangle \\
&=\; \tfrac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta \times C_{\alpha,\beta,\gamma,\delta}\,,
\end{aligned} \tag{S.32}$$

where

$$\begin{aligned}
C_{\alpha,\beta,\gamma,\delta} \;&=\sum_\lambda \langle\alpha|\,\hat{A}_1\,|\lambda\rangle\,\langle\lambda\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle \;+\; \sum_\lambda \langle\beta|\,\hat{A}_1\,|\lambda\rangle\,\langle\alpha\otimes\lambda|\,\hat{B}_2\,|\gamma\otimes\delta\rangle \\
&\quad-\; \sum_\lambda \langle\alpha\otimes\beta|\,\hat{B}_2\,|\lambda\otimes\delta\rangle\ \langle\lambda|\,\hat{A}_1\,|\gamma\rangle \;-\; \sum_\lambda \langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\lambda\rangle\ \langle\lambda|\,\hat{A}_1\,|\delta\rangle \\
&=\; \langle\alpha\otimes\beta|\,\Big(\hat{A}_1(1^{\rm st})\hat{B}_2 \;+\; \hat{A}_1(2^{\rm nd})\hat{B}_2 \;-\; \hat{B}_2\hat{A}_1(1^{\rm st}) \;-\; \hat{B}_2\hat{A}_1(2^{\rm nd})\Big)\,|\gamma\otimes\delta\rangle \\
&=\; \langle\alpha\otimes\beta|\,\Big[\Big(\hat{A}_1(1^{\rm st}) + \hat{A}_1(2^{\rm nd})\Big)\,,\hat{B}_2\Big]\,|\gamma\otimes\delta\rangle \;\equiv\; \langle\alpha\otimes\beta|\,\hat{C}_2\,|\gamma\otimes\delta\rangle\,.
\end{aligned} \tag{S.33}$$

Consequently, $[\hat{A}^{(2)}_{\rm tot}, \hat{B}^{(2)}_{\rm tot}] = \hat{C}^{(2)}_{\rm tot}$. $\mathcal{Q.E.D.}$

Problem **3**(a):

To simplify the $\exp(\xi\hat{a}^\dagger - \xi^*\hat{a})$ in the definition of a coherent state $|\xi\rangle$, we use the product-of-exponentials formula

$$\forall\hat{A},\hat{B}: \quad e^{\hat{A}}e^{\hat{B}} \;=\; \exp\left(\hat{A}+\hat{B} \,+\, \tfrac{1}{2}[\hat{A},\hat{B}] \,+\, \tfrac{1}{12}[(\hat{A}-\hat{B}),[\hat{A},\hat{B}]] \,+\, \cdots\right). \tag{S.34}$$

In particular, for $\hat{A}=\xi\hat{a}^\dagger$, $\hat{B}=-\xi^*\hat{a}$ and $[\hat{A},\hat{B}]=\xi\xi^*$ being a c-number, all the multiple commutators vanish and

$$e^{\xi\hat{a}^\dagger}e^{-\xi^*\hat{a}} \;=\; \exp\left(\xi\hat{a}^\dagger - \xi^*\hat{a} + \tfrac{1}{2}\xi\xi^*\right), \quad \text{exactly.} \tag{S.35}$$

Consequently

$$|\xi\rangle \;\stackrel{\text{def}}{=}\; e^{\xi\hat{a}^\dagger-\xi^*\hat{a}}\,|0\rangle \;=\; e^{-|\xi|^2/2}e^{\xi\hat{a}^\dagger}e^{-\xi^*\hat{a}}\,|0\rangle \;=\; e^{-|\xi|^2/2}e^{\xi\hat{a}^\dagger}\,|0\rangle\,, \tag{S.36}$$

where the last equality follows from $\hat{a}\,|0\rangle=0$ and hence $\exp(-\xi^*\hat{a})\,|0\rangle=|0\rangle$.

Next, $[\hat{a},\hat{a}^\dagger]=1$ implies that for any function $f(\hat{a}^\dagger)$, $[\hat{a},f(\hat{a}^\dagger)]=f'(\hat{a}^\dagger)$. In particular $[\hat{a},e^{\xi\hat{a}^\dagger}]=\xi e^{\xi\hat{a}^\dagger}$, or in other words $(\hat{a}-\xi)\times e^{\xi\hat{a}^\dagger}=e^{\xi\hat{a}^\dagger}\times\hat{a}$. Consequently,

$$(\hat{a}-\xi)\,|\xi\rangle \;=\; e^{-|\xi^2/2}\,(\hat{a}-\xi)\times e^{\xi\hat{a}^\dagger}\,|0\rangle \;=\; e^{-|\xi^2/2}\,e^{\xi\hat{a}^\dagger}\times\hat{a}\,|0\rangle \;=\; 0 \tag{S.37}$$

$\mathcal{Q.E.D.}$

Problem **3**(b):

In the coordinate basis, the annihilation operator $\hat{a}$ acts as

$$\hat{a} \;=\; \frac{\omega m\hat{x}\,+\,i\hat{p}}{\sqrt{2\hbar\omega m}} = \frac{\omega m\hat{x}\,+\,\hbar\partial_x}{\sqrt{2\hbar\omega m}} \tag{S.38}$$

and the condition $\hat{a}\,|\xi\rangle=\xi\,|\xi\rangle$ becomes a first-order differential equation

$$\left(\hbar\frac{d}{dx} \,+\, \omega m\times x \,-\, \sqrt{2\hbar\omega m}\times\xi\right)\psi_\xi(x) \;=\; 0 \tag{S.39}$$

for the wave-function $\psi_\xi(x)$ of the coherent state. This equation has a unique solution (up to overall normalization), namely

$$\psi_\xi(x) \;=\; \text{const} \times \exp\left(\xi\sqrt{\frac{2m\omega}{\hbar}} \times x \;-\; \frac{m\omega}{2\hbar} \times x^2\right), \tag{S.40}$$

or equivalently,

$$\psi_\xi(x) \;=\; \text{const} \times e^{i\bar{p}x/\hbar} \times e^{-m\omega(x-\bar{x})^2/2\hbar}, \tag{S.41}$$

a Gaussian wave-packet with

$$\bar{x} \;=\; \sqrt{\frac{2\hbar}{m\omega}} \times \operatorname{Re}\xi \quad \text{and} \quad \bar{p} \;=\; \sqrt{\frac{\hbar m\omega}{2}} \times \operatorname{Im}\xi. \tag{S.42}$$

Note that the width of the wave packet (S.41) does not depend on ξ, so all coherent states have the same Δx. In particular, since $|\xi = 0\rangle$ is the oscillator's ground state, all coherent states have the same width as the ground state.

Problem **3**(c):

For any *normal-ordered* product of creation and annihilation operators — *i.e.*, a product in which all creation operators are to the right of all annihilation operators — one has

$$\langle\xi|\,(\hat{a}^\dagger)^k(\hat{a})^\ell\,|\xi\rangle \;=\; (\xi^*)^k\xi^\ell, \tag{S.43}$$

simply because $\hat{a}\,|\xi\rangle = \xi\,|\xi\rangle$ and $\langle\xi|\,\hat{a}^\dagger = \xi^*\,\langle\xi|$. In particular, $\langle\xi|\,(\hat{n} = \hat{a}^\dagger\hat{a})\,|\xi\rangle = \xi^*\xi$. On the other hand,

$$\hat{n}^2 \;=\; \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} \;=\; \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} \;+\; \hat{a}^\dagger\hat{a} \;\Longrightarrow\; \langle\xi|\,\hat{n}^2\,|\xi\rangle \;=\; (\xi^*)^2\xi^2 \;+\; \xi^*\xi \;=\; \bar{n}^2 \;+\; \bar{n} \tag{S.44}$$

hence $\Delta n = \sqrt{\langle\hat{n}^2\rangle - \bar{n}^2} = \sqrt{\bar{n}}$.

In a similar manner,

$$\hat{q} \;=\; \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \qquad \hat{q}^2 \;=\; \frac{\hbar}{2m\omega}\Big((\hat{a})^2+(\hat{a}^\dagger)^2+2\hat{a}^\dagger\hat{a}+1\Big), \tag{S.45}$$

hence

$$\langle\xi|\,\hat{q}^2\,|\xi\rangle \;=\; \frac{\hbar}{2m\omega}\big((\xi+\xi^*)^2+1\big) \;=\; \langle\xi|\,\hat{q}\,|\xi\rangle^2 \;+\; \frac{\hbar}{2m\omega}\,. \tag{S.46}$$

Likewise,

$$\langle\xi|\,\hat{p}^2\,|\xi\rangle \;=\; \frac{m\omega\hbar}{2}\big((-i\xi+i\xi^*)^2+1\big) \;=\; \langle\xi|\,\hat{p}\,|\xi\rangle^2 \;+\; \frac{m\omega\hbar}{2}\,.$$

Altogether, this gives us for any coherent state

$$\Delta q \;=\; \sqrt{\frac{\hbar}{2m\omega}}\,, \quad \Delta p \;=\; \sqrt{\frac{m\omega\hbar}{2}}\,, \quad \Delta q\Delta p \;=\; \frac{\hbar}{2}\,. \tag{S.47}$$

$\mathcal{Q.E.D.}$

Problem **3**(d):
According to eq. (10),

$$|\xi(t)\rangle \;=\; e^{-|\xi|^2/2}\,e^{\xi(t)\hat{a}^\dagger}\,|0\rangle\,, \tag{S.48}$$

and only the second factor here depends on time. Indeed, $e^{-|\xi|^2/2}=\text{const}$ for $\xi(t)=\xi_0 e^{-i\omega t}$, and $|0\rangle$ is time-independent because we work in the Schrödinger picture. In this picture, the $\hat{a}^\dagger$ operator is also time independent, hence

$$\frac{d}{dt}\,e^{\xi\hat{a}^\dagger} \;=\; \frac{d\xi}{dt}\,\hat{a}^\dagger\times e^{\xi\hat{a}^\dagger} \;=\; -i\omega\xi\,\hat{a}^\dagger\times e^{\xi\hat{a}^\dagger}, \tag{S.49}$$

and therefore

$$\frac{d}{dt}\,|\xi\rangle \;=\; -i\omega\xi\,\hat{a}^\dagger\,|\xi\rangle \;=\; -i\omega\,\hat{a}^\dagger\hat{a}\,|\xi\rangle \tag{S.50}$$

where the second equality follows from $\xi\,|\xi\rangle=\hat{a}\,|\xi\rangle$. Consequently,

$$i\hbar\frac{d}{dt}\,|\xi(t)\rangle \;=\; \hbar\omega\hat{a}^\dagger\hat{a}\,|\xi(t)\rangle \;\equiv\; \hat{H}\,|\xi(t)\rangle \tag{S.51}$$

— the coherent state $|\xi(t)\rangle$ satisfies the Schrödinger equation. $\mathcal{Q.E.D.}$

Problem $\mathbf{3}$(e):

In question $\mathbf{3}$(a) we saw that $[\hat{a},\hat{a}^\dagger]=1$ implies $e^{\xi\hat{a}^\dagger}\hat{a}=(\hat{a}-\xi)e^{\xi\hat{a}^\dagger}$ for any c-number ξ. Iterating this identity gives us $e^{\xi\hat{a}^\dagger}f(\hat{a})=f(\hat{a}-\xi)e^{\xi\hat{a}^\dagger}$ for any function $f(\hat{a})$ of the annihilation operator, and in particular

$$\exp(\xi\hat{a}^\dagger)\times\exp(\eta^*\hat{a}) \;=\; \exp(\eta^*(\hat{a}-\xi))\times\exp(\xi\hat{a}^\dagger) \;=\; \exp(-\eta^*\xi)\times\exp(\eta^*\hat{a})\times\exp(\xi\hat{a}^\dagger). \quad \text{(S.52)}$$

Consequently, the quantum overlap of the coherent states $|\xi\rangle$ and $\langle\eta|$ is

$$\begin{aligned}\langle\eta|\xi\rangle \;&=\; e^{-|\eta|^2/2}e^{-|\xi|^2/2}\times\langle 0|\exp(\eta^*\hat{a})\,\exp(\xi\hat{a}^\dagger)\,|0\rangle\\ &=\; e^{-|\eta|^2/2}e^{-|\xi|^2/2}e^{+\eta^*\xi}\,\langle 0|\exp(\xi\hat{a}^\dagger)\,\exp(\eta^*\hat{a})\,|0\rangle\\ &=\; \exp\left(-\tfrac{1}{2}|\eta|^2-\tfrac{1}{2}|\xi|^2+\eta^*\xi\right)\times 1\end{aligned} \quad \text{(S.53)}$$

because $e^{\eta^*\hat{a}}\,|0\rangle=|0\rangle$, $\langle 0|\,e^{\xi\hat{a}^\dagger}=\langle 0|$, and $\langle 0|0\rangle=1$. In terms of the probability overlap,

$$\left|\langle\eta|\xi\rangle\right|^2 \;=\; e^{-|\eta-\xi|^2}. \quad \text{(S.54)}$$

Problem $\mathbf{3}$(f):

Generalization of coherent states to multi-oscillatory systems and further to the creation / annihilation fields is completely straightforward:

$$|\text{coherent}\rangle \;\stackrel{\text{def}}{=}\; \exp\bigl(\hat{F}^\dagger-\hat{F}\bigr)\,|0\rangle \;=\; e^{-\bar{N}/2}\,e^{\hat{F}^\dagger}\,|0\rangle \quad \text{(S.55)}$$

where

$$\hat{F}^\dagger \;=\; \xi\hat{a}^\dagger \;\to\; \sum_\alpha \xi_\alpha\hat{a}^\dagger_\alpha \;\to\; \int d^3\mathbf{x}\,\Phi(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{x}). \quad \text{(S.56)}$$

Similar to the single-oscillator theory, $\bigl(\hat{\Psi}(\mathbf{x})-\Phi(\mathbf{x})\bigr)e^{\hat{F}^\dagger}=e^{\hat{F}^\dagger}\hat{\Psi}^\dagger(\mathbf{x})$, hence

$$\hat{\Psi}(\mathbf{x})\,|\Phi\rangle \;=\; \Phi(\mathbf{x})\,|\Phi\rangle\,. \quad \text{(S.57)}$$

Problem $\mathbf{3}$(g):

Using eq. (S.57) and its hermitian conjugate, we have

$$\langle\Phi|\,\hat{\Psi}^\dagger(\mathbf{x}_1)\cdots\hat{\Psi}^\dagger(\mathbf{x}_k)\hat{\Psi}(\mathbf{y}_1)\cdots\hat{\Psi}(\mathbf{y}_\ell)\,|\Phi\rangle \;=\; \Phi^*(\mathbf{x}_1)\cdots\Phi^*(\mathbf{x}_k)\Phi(\mathbf{y}_1)\cdots\Phi(\mathbf{y}_\ell) \tag{S.58}$$

for any *normal-ordered* product of the quantum fields. Specifically, for the particle-number operator $\hat{N}$ we have eq. (10), while for its square — whose normal-ordered form

$$\hat{N}^2 \;=\; \iint d^3\mathbf{x}\,d^3\mathbf{y}\;\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{y})\hat{\Psi}(\mathbf{x})\hat{\Psi}(\mathbf{y}) \;+\; \int d^3\mathbf{x}\,\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}(\mathbf{x}) \tag{S.59}$$

generalizes eq. (S.44) — we have

$$\langle\Phi|\,\hat{N}^2\,|\Phi\rangle \;=\; \iint d^3\mathbf{x}\,d^3\mathbf{y}\;\Phi^*(\mathbf{x})\Phi^*(\mathbf{y})\Phi(\mathbf{x})\Phi(\mathbf{y}) \;+\; \int d^3\mathbf{x}\,\Phi^*(\mathbf{x})\Phi(\mathbf{x}) \;=\; \langle\Phi|\,\hat{N}\,|\Phi\rangle^2 \;+\; \langle\Phi|\,\hat{N}\,|\Phi\rangle\,, \tag{S.60}$$

and hence $\Delta N = \sqrt{\bar{N}}$. $\quad\mathcal{Q.E.D.}$

Problem $\mathbf{3}$(h):

First of all, if $\Phi(\mathbf{x},t)$ satisfies the classical field equation — which looks exactly like a one-particle Schrödinger equation — then $\bar{N}$ remains constant. (This is undergraduate-level QM.) Also, in the Schrödinger picture of the QFT,

$$\frac{d}{dt}\,e^{\hat{F}^\dagger} \;=\; \frac{d\hat{F}^\dagger}{dt}\,e^{\hat{F}^\dagger} \;=\; \left[\int d^3\mathbf{x}\,\frac{\partial\Phi(\mathbf{x},t)}{\partial t}\,\hat{\Psi}^\dagger(\mathbf{x})\right]\,e^{\hat{F}^\dagger} \tag{S.61}$$

thanks to mutual commutativity of the creation fields. Consequently, exactly as in question (c),

$$\begin{aligned}
i\hbar\frac{d}{dt}\Big(|\Phi\rangle = e^{-\bar{N}/2}e^{\hat{F}^\dagger}\,|0\rangle\Big) \;&=\; \left[\int d^3\mathbf{x}\,i\hbar\frac{\partial\Phi(\mathbf{x},t)}{\partial t}\,\hat{\Psi}^\dagger(\mathbf{x})\right]|\Phi\rangle \\
&\langle\!\langle\text{using the classical field equation for }\Phi\rangle\!\rangle \\
&=\; \left[\int d^3\mathbf{x}\,\left(\left(\frac{-\hbar^2}{2M}\nabla^2+V(\mathbf{x})\right)\Phi(\mathbf{x})\right)\,\hat{\Psi}^\dagger(\mathbf{x})\right]|\Phi\rangle \\
&\langle\!\langle\text{using eq. (9)}\rangle\!\rangle \\
&=\; \left[\int d^3\mathbf{x}\,\hat{\Psi}^\dagger(\mathbf{x})\left(\frac{-\hbar^2}{2M}\nabla^2+V(\mathbf{x})\right)\hat{\Psi}(\mathbf{x})\right]|\Phi\rangle \\
&=\; \hat{H}\,|\Phi\rangle\,.
\end{aligned} \tag{S.62}$$

$\mathcal{Q.E.D.}$

Problem **3**(i):

Generalizing (e) to multi-oscillatory systems is completely straightforward:

$$|\langle\eta|\xi\rangle|^2 \;=\; \prod_\alpha e^{-|\xi_\alpha-\eta_\alpha|^2} \;=\; \exp\left(-\sum_\alpha|\xi_\alpha-\eta_\alpha|^2\right)$$

or for the field theory,

$$|\langle\Phi_1|\Phi_2\rangle|^2 \;=\; \exp\left(\int d^3\mathbf{x}\; |\Phi_1(\mathbf{x})-\Phi_2(\mathbf{x})|^2\right), \tag{S.63}$$

which is exponentially small for any macroscopic $\delta\Phi(\mathbf{x})=\Phi_1(x)-\Phi_2(x)$. Indeed, a *macroscopic* difference between two coherent states means (by definition) that $\delta\Phi$ affects a large number of particles, $\int|\delta\Phi|^2\gg 1$, which makes for an *exponentially* tiny overlap (S.63).