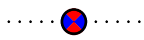
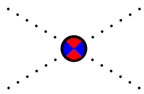
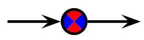
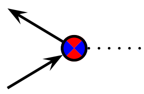
Problem **1** — the textbook problem **10.2**(b):

In part (a) of this problem (see solutions to the previous homework set #16), we showed that the Yukawa theory needs 6 counterterms δ_m^ϕ, δ_Z^ϕ, δ_m^ψ, δ_Z^ψ, δ_g, and δ_λ, which lead to four counterterm vertices:

 $= -i\delta_m^\phi + ip^2\,\delta_Z^\phi\,,$

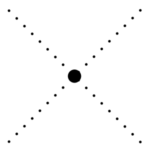
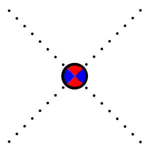
 $= -i\delta_\lambda\,,$

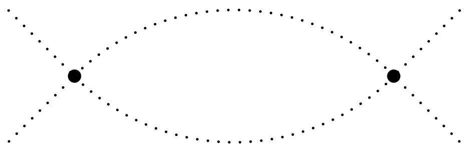
 $= -i\delta_m^\psi + i\,\not{p}\,\delta_Z^\psi\,,$ (S.1)

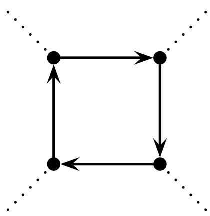
 $= -\delta_g\gamma^5$

In this part (b) we shall calculate the infinite parts of all the counterterms.

Let's start with the δ_λ counterterms which cancels the divergences of the four-scalar 1PI amplitude $\mathcal{V}(k_1,k_2,k_3,k_4)$. At the one-loop level of analysis, we have the following Feynman diagrams:

$i\mathcal{M}^{1\,\text{loop}}(k_1,k_2,k_3,k_4)$ =  + 

+  + two similar (S.2)

+  + five similar.

The last diagram here yields

$$-\int\frac{d^4p_1}{(2\pi)^4}\,\text{Tr}\left\{(-g\gamma^5)\,\frac{i}{\not{p}_1-M+i0}\,(-g\gamma^5)\,\frac{i}{\not{p}_2-M+i0}\,(-g\gamma^5)\,\frac{i}{\not{p}_3-M+i0}\,(-g\gamma^5)\,\frac{i}{\not{p}_4-M+i0}\right\} \tag{S.3}$$

where

$$p_2 = p_1 + k_1\,, \qquad p_3 = p_2 + k_2, \qquad p_4 = p_3 + k_3, \quad \text{and} \quad p_1 = p_4 + k_4\,;$$

there are five similar diagrams related by permutations of the external momenta k_1, k_2, k_3, k_4. For generic values of these momenta, the integral (S.3) is quite complicated, but its divergence is k-independent and hence may be evaluated for any particular choice of k_i we find convenient. Clearly, the simplest set of k_i is $k_1 = k_2 = k_3 = k_4 = 0$; this is off-shell, but that's OK. Consequently, the integral (S.3) becomes

$$\begin{aligned} i\mathcal{V}^{\psi\,\text{loop}}(0,0,0,0) &= -\int\frac{d^4p_1}{(2\pi)^4}\,\text{tr}\left[\left((-g\gamma^5)\,\frac{i}{\not p - M + i0}\right)^4\right] \\ &= -g^4\int\frac{d^4p_1}{(2\pi)^4}\,\frac{\text{tr}\left[\left(\gamma^5(\not p + M)\right)^4\right]}{(p^2 - M^2 + i0)^4} \\ &= \int\frac{d^4p_1}{(2\pi)^4}\,\frac{-4g^4}{(p^2 - M^2 + i0)^2} \end{aligned} \tag{S.4}$$

where the last equality follows from

$$\left(\gamma^5(\not p + M)\right)^2 \;=\; \gamma^5(\not p + M)\gamma^5(\not p + M) \;=\; (-\not p + M)(\not p + M) \;=\; -p^2 \,+\, M^2 \tag{S.5}$$

and hence

$$\text{tr}\left[\left(\gamma^5(\not p + M)\right)^4\right] \;=\; 4(p^2 - M^2)^2. \tag{S.6}$$

Evaluating the integral on the last line of eq. (S.4) using dimensional regularization, we obtain

$$\mathcal{V}_{\psi\,\text{loop}}(k_1 = k_2 = k_3 = k_4 = 0) \;=\; \frac{-4g^4}{16\pi^2}\left(\frac{1}{\bar\epsilon} + \log\frac{\mu^2}{M^2}\right) \tag{S.7}$$

where

$$\frac{1}{\bar\epsilon} \stackrel{\text{def}}{=} \frac{1}{\epsilon} - \gamma_E \,+\, \log(4\pi). \tag{S.8}$$

This notation is common in dimensional regularization: because the $1/\epsilon$ divergence is usually accompanied by the finite $-\gamma_E + \log(4\pi)$ constant, it's convenient to combine them into a single term denoted $1/\bar\epsilon$.

It remains to multiply the amplitude (S.7) by 6 (for six similar diagrams) and add contributions of the other diagrams (S.2). The latter diagrams have been evaluated in class in the context of the scalar $\lambda\Phi^4$ theory, thus to order $O(\lambda^2 \text{ or } g^4)$,

$$\mathcal{V}(k_1 = k_2 = k_3 = k_4 = 0) \;=\; -\lambda \;-\; \delta_\lambda \;+\; \frac{3\lambda^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\frac{\mu^2}{m^2}\right) \;-\; \frac{24g^4}{16\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\frac{\mu^2}{M^2}\right). \quad \text{(S.9)}$$

The renormalization condition for the physical λ coupling is the on-shell four-particle amplitude $\mathcal{M}(\text{threshold}) = -\lambda$, or in other words $\mathcal{V} = -\lambda$ when all external momenta are on shell and at the threshold ($s = 4m^2$, $t = u = 0$). At other values of external momenta, we should have

$$\mathcal{V}(k_1, k_2, k_3, k_4) \;=\; -\lambda \;-\frac{\lambda^2}{32\pi^2}\times\text{finite} \;-\; \frac{4g^4}{16\pi^2}\times\text{finite} \;+\; \text{higher loop orders.} \quad \text{(S.10)}$$

Comparing this formula with eq. (S.9) gives us

$$\delta_\lambda^{1\,\text{loop}} \;=\; \frac{3\lambda^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\frac{\mu^2}{m^2} + \text{finite}\right) \;-\; \frac{24g^4}{16\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\frac{\mu^2}{M^2} + \text{finite}\right). \quad \text{(S.11)}$$

As promised last week, fermionic loops provide for $\delta_\lambda \neq 0$ even if were to start from $\lambda = 0$.

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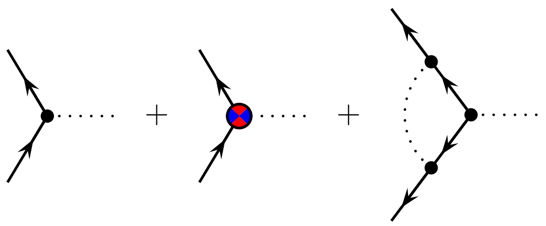
Next, we want to calculate the δ_g counterterm, so let us consider the $\Phi\overline{\Psi}\gamma^5\Psi$ vertex correction. By analogy with the QED vertex, we denote $\Gamma^{(5)}(p', p)$ the 1PI amplitude for two fermions of respective momenta p and p' and one pseudoscalar of momentum $k = p' - p$. At the one-loop level of analysis,

$-\Gamma^{(5)}(p', p) \;=$



(S.12)

$$= \; -g\gamma^5 \;-\; \delta_g\gamma^5$$

$$+ \int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2 - m^2 + i0}\times(-g\gamma^5)\,\frac{i}{\not{p}' + \not{q} - M + i0}\,(-g\gamma^5)\,\frac{i}{\not{p} + \not{q} - M + i0}\,(-g\gamma^5).$$

As in the previous calculation, the loop integral here diverges logarithmically, and the divergent part does not depend on the external momenta. Consequently, we may calculate this divergence for any values of p, p', and $k = p' - p$ we like, for example $p = p' = k = 0$ which makes for a much simpler integral. Indeed, for zero external momenta, the fermionic line becomes

$$\begin{aligned}(-g\gamma^5)\,\frac{i}{0+\not{q} - M + i0}\,(-g\gamma^5)\,\frac{i}{0+\not{q} - M + i0}\,(-g\gamma^5) \;&=\; g^3\,\frac{\gamma^5(\not{q}+M)\gamma^5(\not{q}+M)\gamma^5}{(q^2-M^2+i0)^2} \\ &=\; g^3\,\frac{-\gamma^5}{q^2-M^2+i0}\end{aligned} \tag{S.13}$$

where the second equality follows from eq. (S.5). Consequently, the loop integral in eq. (S.12) becomes

$$\begin{aligned}-ig^3\gamma^5\times\int\!\frac{d^4q}{(2\pi)^4}\,\frac{1}{(q^2-m^2+i0)(q^2-M^2+i0)} \;&=\; +g^3\gamma^5\times\int\!\frac{d^4q_E}{(2\pi)^4}\,\frac{1}{(q^2+m^2)(q^2+M^2)} \\ \langle\!\langle\text{using dimensional regularization}\rangle\!\rangle \qquad & \\ &=\; +\frac{g^3\gamma^5}{16\pi^2}\left(\frac{1}{\bar\epsilon}\;+\;\int\limits_0^1\!dx\,\log\frac{\mu^2}{xM^2+(1-x)m^2}\right)\end{aligned} \tag{S.14}$$

and hence to the order g^3,

$$\Gamma^{(5)}(p'=p=0) \;=\; -g\gamma^5 \;-\; \delta_g\gamma^5 \;+\; \frac{g^3\gamma^5}{16\pi^2}\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{M^2}\;+\;\text{finite}\right). \tag{S.15}$$

And since the divergent part is momentum independent, it follows that for any external momenta,

$$\Gamma^{(5)}(p',p) \;=\; -g\gamma^5 \;-\; \delta_g\gamma^5 \;+\; \frac{g^3\gamma^5}{16\pi^2}\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{M^2}\;+\;\text{finite function of}(p',p)\right) \;+\; O(g^5\text{ or }g^3\lambda). \tag{S.16}$$

In class, I have not explained the renormalization condition for the Yukawa coupling g, but it's clear that such condition should have form $\Gamma^{(5)} = -g\gamma^5$ for the on-shell fermions and some particular value of the pseudoscalar's q^2, for example $q^2 = 0$ or on-shell $q^2 = m^2$ (allowed for $m \geq 2M$). In light of eq. (S.16), this means

$$\delta_g^{1\,\text{loop}} \;=\; \frac{g^3}{16\pi^2}\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{M^2}\;+\;\text{finite}\right) \tag{S.17}$$

where the finite part depends on the specific renormalization condition (and in general is a painfully complicated function of the m/M mass ratio), but the infinite part is clear and unambiguous.

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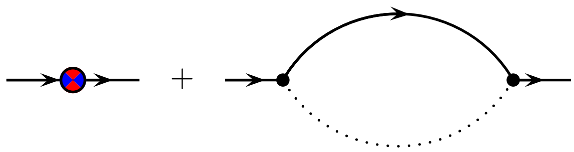
Our next targets are the fermion's mass and kinetic energy counterterms δ_M^ψ and δ_Z^ψ. At the one-loop level of analysis, the Dirac field's 1PI two-point Green's function is



$$-i\Sigma_\psi^{\text{1 loop order}}(\not{p}) = \ldots$$

$$= -i\delta_M^\psi + i\delta_Z^\psi \not{p} + \int\frac{d^4k}{(2\pi)^4}\,\frac{i}{q^2-m^2+i0}\times(-g\gamma^5)\,\frac{i}{\not{p}+\not{q}-M+i0}\,(-g\gamma^5). \tag{S.18}$$

This time, we cannot set $p=0$ so we must be more careful. Let us re-write the integrand of the loop integral as

$$-g^2\,\frac{\mathcal{N}}{\mathcal{D}} \tag{S.19}$$

where

$$\mathcal{N} = \gamma^5(\not{q}+\not{p}+M)\gamma^5 \quad\text{and}\quad \mathcal{D} = \Big(q^2-m^2+i0\Big)\times\Big((p+q)^2-M^2+i0\Big). \tag{S.20}$$

Using the Feynman's parameter trick we may simplify the denominator as

$$\begin{aligned}\frac{1}{\mathcal{D}} &= \int_0^1\!dx\,\frac{1}{\big[(1-x)(q^2-m^2)\,+\,x((p+q)^2-M^2)\,+\,i0\big]^2}\\ &= \int_0^1\!dx\,\frac{1}{\big[\ell^2\,-\,\Delta\,+\,i0\big]^2}\end{aligned} \tag{S.21}$$

where

$$\ell = q + xp \quad\text{and}\quad \Delta = xM^2 + (1-x)m^2 - x(1-x)p^2. \tag{S.22}$$

As usual, we take the $\int dx$ integral after integrating over momentum, which allows us to shift the momentum variable from p^μ to ℓ^μ, thus

$$\Sigma_\psi^{1\,\text{loop}}(\not{p}) \;=\; -\,ig^2\int\limits_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}}{[\ell^2-\Delta+i0]^2}\,. \tag{S.23}$$

In terms of the shifted loop momentum ℓ, the numerator becomes

$$\mathcal{N} \;=\; \gamma^5\Big(\not{\ell}+(1-x)\,\not{p}+M\Big)\gamma^5 \;=\; M\;-\;(1-x)\,\not{p}-\;\not{\ell}, \tag{S.24}$$

where the lat term $\not{\ell}$ does not contribute to the momentum integral because it's odd under the $\ell\to-\ell$ symmetry, thus

$$\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\not{\ell}}{[\ell^2-\Delta+i0]^2} \;=\; 0 \tag{S.25}$$

and hence

$$\Sigma_\psi^{1\,\text{loop}}(\not{p}) \;=\; -ig^2\int\limits_0^1 dx\,[M-(1-x)\,\not{p}]\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2-\Delta+i0)^2}\,. \tag{S.26}$$

Note that although the two-fermion amplitude Σ_ψ has superficial degree of divergence $D=+1$, the leading linear divergence (S.25) vanishes by Lorentz symmetry, and the remaining momentum integral (S.26) has only the sub-leading logarithmic UV divergence. Evaluating this integral by going to the Euclidean momentum space and using dimensional regularization, we obtain

$$\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2-\Delta+i0)^2} \;=\; \frac{i}{16\pi^2}\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{\Delta}\right), \tag{S.27}$$

and therefore

$$\Sigma_\psi^{1\,\text{loop}}(\not{p}) \;=\; \delta_M^\psi\;-\;\delta_Z^\psi\,\not{p}+\frac{g^2}{16\pi^2}\int\limits_0^1 dx\,[M-(1-x)\,\not{p}]\,\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{(1-x)m^2+xM^2-x(1-x)p^2}\right). \tag{S.28}$$

The renormalization conditions for the fermion's propagator correction $\Sigma^\psi(\not{p})$ are

$$\Sigma\bigg|_{\not{p}\,=\,M} \;=\; 0 \quad\text{and}\quad \frac{d\Sigma}{d\,\not{p}}\bigg|_{\not{p}\,=\,M} \;=\; 0. \tag{S.29}$$

In light of eq. (S.28), the second condition (S.29) becomes

$$\begin{aligned}
\delta_Z^{\psi}[1\,\text{loop}] &= \frac{g^2}{16\pi^2}\,\frac{\partial}{\partial\,\not{p}}\int\limits_0^1 dx\,[M-(1-x)\not{p}]\,\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{(1-x)m^2+xM^2-x(1-x)p^2}\right)\Bigg|_{\not{p}=M}\\
&= \frac{g^2}{16\pi^2}\int\limits_0^1 dx\left[(x-1)\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{x^2M^2+(1-x)m^2}\right)\,+\,\frac{2x^2(1-x)M^2}{x^2M^2+(1-x)m^2}\right]\\
&= -\frac{g^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{M^2}\,+\,\text{finite}\right).
\end{aligned}\tag{S.30}$$

At the same time, the first condition (S.29) implies

$$\begin{aligned}
&\delta_M^{\psi}[1\,\text{loop}]\;-\;M\delta_Z^{\psi}[1\,\text{loop}]\\
&\quad= -\frac{g^2}{16\pi^2}\int\limits_0^1 dx\,[M-(1-x)\not{p}]\,\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{(1-x)m^2+xM^2-x(1-x)p^2}\right)\Bigg|_{\not{p}=M}\\
&\quad= -\frac{g^2}{16\pi^2}\int\limits_0^1 dx\,xM\times\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{x^2M^2+(1-x)m^2}\right)\\
&\quad= -\frac{g^2M}{32\pi^2}\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{M^2}\,+\,\text{finite}\right)
\end{aligned}\tag{S.31}$$

and consequently

$$\delta_M^{\psi}[1\,\text{loop}]\;=\;-\frac{g^2M}{16\pi^2}\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{M^2}\,+\,\text{finite}\right).\tag{S.32}$$

Note that similarly to QED, the fermionic mass counterterm in the Yukawa theory is proportional to the mass itself and diverges logarithmically rather than linearly in the UV cutoff (*cf.* integral (S.26) prior to dimensional regularization). As in QED, this behavior is due an additional symmetry the Yukawa theory acquires when the fermion mass vanishes. Specifically, for $M=0$ we have a *discrete chiral symmetry*

$$\Psi(x)\;\to\;\gamma^5\Psi(x),\quad\overline{\Psi}(x)\;\to\;-\overline{\Psi}(x)\gamma^5,\quad\Phi(x)\;\to\;-\Phi(x).\tag{S.33}$$

Unlike the gauge coupling in QED, the pseudoscalar Yukawa coupling does not respect continuous chiral transforms $\Psi(x)\to\exp(i\alpha\gamma^5)\Psi(x)$, but the discrete symmetry is sufficient for preventing the massless Yukawa theory from developing a mass shift via loop corrections.

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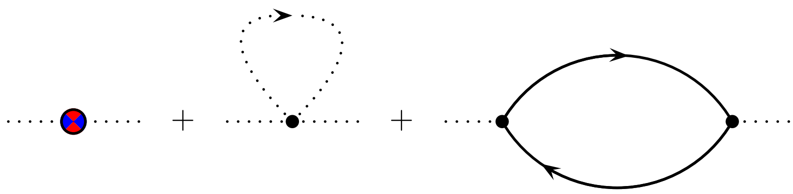
Finally, consider the boson's mass and kinetic energy counterterms δ_M^ϕ and δ_Z^ϕ. At the one-loop level of analysis, the pseudoscalar field's 1PI two-point Green's function is

$$-i\Sigma_\phi^{1\,\text{loop}}(k^2) \;=$$



$$\begin{aligned} &= -i\delta_m^\phi \;+\; i\delta_Z^\phi\,k^2 \;+\; \frac{i\lambda m^2}{32\pi^2}\left(\frac{1}{\bar\epsilon} \;+\; 1 \;+\; \log\frac{\mu^2}{m^2}\right) \\ &\quad - \int\frac{d^4p}{(2\pi)^4}\;\text{tr}\left(\frac{i}{\not p - M + i0}\,(-g\gamma^5)\,\frac{i}{\not p + \not k - M + i0}\,(-g\gamma^5)\right). \end{aligned} \tag{S.34}$$

Again, we re-write the fermionic loop integral as

$$+\,g^2\int\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}}{\mathcal{D}} \tag{S.35}$$

where the denominator is the usual

$$\mathcal{D} \;=\; \Big(p^2 - M^2 + i0\Big)\times\Big((p+k)^2 - M^2 + i0\Big) \tag{S.36}$$

and hence

$$\begin{aligned} \frac{1}{\mathcal{D}} \;&=\; \int_0^1 dx\,\frac{1}{[\ell^2 - \Delta + i0]^2} \\ \text{for } \ell \;&=\; p \;+\; kx \\ \text{and } \Delta \;&=\; M^2 \;-\; x(1-x)k^2, \end{aligned} \tag{S.37}$$

and the numerator is

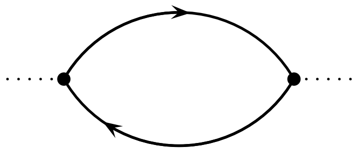
$$\begin{aligned} \mathcal{N} \;&=\; \text{tr}\big[(\not p + M)\gamma^5(\not p + \not k + M)\gamma^5\big] \\ &=\; \text{tr}\big[(M + \not p)(M - \not p - \not k)\big] \\ &=\; 4M^2 \;-\; 4p(p+k) \\ &=\; 4M^2 \;-\; 4(\ell - xk)(\ell + k - xk) \\ &=\; 4M^2 \;-\; 4\ell^2 \;+\; 4x(1-x)k^2 \;-4(1-2x)\ell\cdot k. \end{aligned} \tag{S.38}$$

Again, the last term here is odd with respect to $\ell \to -\ell$ and hence does not contribute to the $\int d^4\ell$

integral. Effectively,

$$\mathcal{N} \;\cong\; 4M^2 \;-\; 4\ell^2 \;+\; 4x(1-x)k^2$$

and hence the integral (S.35) becomes



$$\begin{aligned} &= 4g^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\, \frac{M^2 + x(1-x)k^2 - \ell^2}{(\ell^2 - \Delta + i0)^2} \\ &= 4ig^2 \int_0^1 dx \int \frac{d^4\ell_E}{(2\pi)^4}\, \frac{2M^2 - \Delta + \ell_E^2}{(\ell_E^2 + \Delta)^2}\,. \end{aligned} \tag{S.39}$$

In four dimensions, the momentum integral (S.39) diverges quadratically. Hence, in dimensional regularization, we need to analytically continue from $D = 4$ Euclidean dimensions down to $D < 2$, evaluate the integral for $D < 2$, and only then continue back to $D = 4 - 2\epsilon$. Thus, working in the Euclidean momentum space, we have

$$\begin{aligned} \int \frac{d^4\ell}{(2\pi)^4}\, \frac{2M^2 - \Delta + \ell^2}{(\ell^2 + \Delta)^2} \;&\longrightarrow\; \mu^{4-D} \int \frac{d^D\ell}{(2\pi)^D}\, \frac{2M^2 - \Delta + \ell^2}{(\ell^2 + \Delta)^2} \\ &= \mu^{4-D} \int \frac{d^D\ell}{(2\pi)^D} \int_0^\infty dt\, t\, e^{-t\Delta} \left(2M^2 - \Delta - \frac{\partial}{\partial t}\right) e^{-t\ell^2} \\ &= \mu^{4-D} \int_0^\infty dt\, t\, e^{-t\Delta} \left(2M^2 - \Delta - \frac{\partial}{\partial t}\right) \left[\int \frac{d^D\ell_E}{(2\pi)^D}\, e^{-t\ell_E^2} \;=\; (4\pi t)^{-D/2}\right] \\ &= \frac{\mu^{4-D}}{(4\pi)^{D/2}} \int_0^\infty dt\, t\, e^{-t\Delta} \left((2M^2 - \Delta) t^{-(D/2)} \;+\; \tfrac{D}{2}\, t^{-(D/2)-1}\right) \\ &= \frac{\mu^{4-D}}{(4\pi)^{D/2}} \left((2M^2 - \Delta)\Gamma(2 - \tfrac{D}{2})\Delta^{(D/2)-2} \;+\; \tfrac{D}{2}\Gamma(1 - \tfrac{D}{2})\Delta^{(D/2)-1}\right) \\ &\langle\!\langle \text{now take } D = 4 - 2\epsilon \rangle\!\rangle \\ &= \frac{1}{16\pi^2}\, \Gamma(\epsilon) \left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon} \left(2M^2 \;-\; \Delta \;+\; \frac{2-\epsilon}{\epsilon - 1}\Delta\right) \\ &\underset{\epsilon\to 0}{\longrightarrow}\; \frac{1}{16\pi^2} \left[(2M^2 - 3\Delta)\left(\frac{1}{\epsilon} \;+\; \log\frac{\mu^2}{\Delta}\right) \;-\; \Delta\right]. \end{aligned} \tag{S.40}$$

Consequently,

$$\begin{aligned}\Sigma_\phi^{1\,\text{loop}}(k^2) \;&=\; \delta_m^\phi \;-\; \delta_Z^\phi\, k^2 \;-\; \frac{\lambda m^2}{32\pi^2}\left(\frac{1}{\epsilon}\,+\,1\,+\,\log\frac{\mu^2}{m^2}\right)\\ &\quad -\;\frac{g^2}{4\pi^2}\int\limits_0^1\!dx\left[(2M^2-3\Delta)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{\Delta}\right)\;-\;\Delta\right].\end{aligned}\tag{S.41}$$

Similarly to the fermion's propagator correction Σ_ψ discussed above, the renormalization conditions for a scalar or a pseudoscalar field are

$$\Sigma_\phi\Big|_{k^2=m^2} \;=\; 0 \qquad\text{and}\qquad \frac{\partial\Sigma_\phi}{\partial k^2}\Big|_{k^2=m^2} \;=\; 0.\tag{S.42}$$

Therefore, in light of eq. (S.41),

$$\begin{aligned}\delta_Z^\phi[1\,\text{loop}] \;&=\; -\frac{g^2}{4\pi^2}\frac{\partial}{\partial k^2}\int\limits_0^1\!dx\left[(2M^2-3\Delta)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{\Delta}\right)\;-\;\Delta\right]_{k^2=m^2}\\ &=\; +\frac{g^2}{4\pi^2}\int\limits_0^1\!dx\,x(1-x)\times\frac{\partial}{\partial\Delta}\left((2M^2-3\Delta)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{\Delta}\right)\;-\;\Delta\right)\Big|_{k^2=m^2}\\ &=\; -\frac{g^2}{4\pi^2}\int\limits_0^1\!dx\,x(1-x)\left[\frac{3}{\bar\epsilon}\;+\;3\log\frac{\mu^2}{M^2-x(1-x)m^2}\;+\;\frac{2x(1-x)k^2}{M^2-x(1-x)m^2}\right]\\ &=\; -\frac{g^2}{8\pi^2}\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{M^2}\;+\;\text{finite}\right).\end{aligned}\tag{S.43}$$

Likewise,

$$\begin{aligned}&\delta_m^\phi[1\,\text{loop}] \;-\; m^2\,\delta_Z^\phi[1\,\text{loop}] \;-\; \frac{\lambda m^2}{32\pi^2}\left(\frac{1}{\epsilon}\,+\,1\,+\,\log\frac{\mu^2}{m^2}\right)\\ &\quad=\; -\frac{g^2}{4\pi^2}\int\limits_0^1\!dx\left[(2M^2-3\Delta)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{\Delta}\right)\;-\;\Delta\right]_{k^2=m^2}\\ &\quad=\; -\frac{g^2}{4\pi^2}\int\limits_0^1\!dx\left[\Big(3x(1-x)m^2-M^2\Big)\times\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2-x(1-x)m^2}\right)+\;\text{finite}\right]\\ &\quad=\; -\frac{g^2}{4\pi^2}\left((\tfrac{1}{2}m^2-M^2)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2}\right)+\;\text{finite}\right]\end{aligned}\tag{S.44}$$

and hence

$$\delta_m^\phi[1\,\text{loop}] \;=\; \left[\frac{\lambda m^2}{32\pi^2} \;+\; \frac{g^2M^2}{4\pi^2}\right] \times \left(\frac{1}{\bar\epsilon} \;+\; \log\frac{\mu^2}{M^2}\right) \;+\; \text{finite}. \tag{S.45}$$

Note that unlike the other counterterms of the Yukawa theory, the pseudoscalar mass correction δ_m^ϕ diverges quadratically rather than logarithmically. The dimensional regularization however does not see the quadratic divergence itself, all it sees is the sub-leading logarithmic divergence accompanying the quadratic divergence. Thus, in terms of a different UV cutoff, eq. (S.45) means

$$\delta_m^\phi[1\,\text{loop}] \;=\; (\text{unknown})\times\Lambda^2 \;+\; \left[\frac{\lambda m^2}{32\pi^2} \;+\; \frac{g^2M^2}{4\pi^2}\right] \times \log\frac{\Lambda^2}{M^2} \;+\; \text{finite}, \tag{S.46}$$

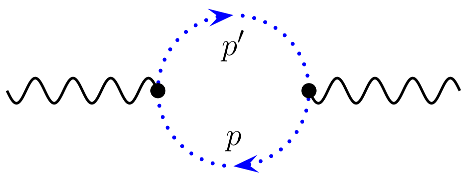
where the coefficient of the leading Λ^2 divergence depends on the cutoff's details — such as the exact definition of Λ^2 for each cutoff. FYI, for the Wilson's hard-edge cutoff

$$\delta_m^\phi[1\,\text{loop}] \;=\; -\frac{\lambda}{32\pi^2}\left(\Lambda^2 \;-\; m^2\log\frac{\Lambda^2}{m^2}\right) \;-\; \frac{g^2}{4\pi^2}\left(\Lambda^2 \;-\; M^2\log\frac{\Lambda^2}{M^2}\right) \;+\text{finite}. \tag{S.47}$$

⋆ ⋆ ⋆ ⋆ ⋆

Problem **2**(a):

Let us start with the first diagram



(S.48)

Direct evaluation of the Feynman rules gives us

$$i\Sigma^{\mu\nu}_{(1)}(k) \;=\; \int\!\frac{d^4p}{(2\pi)^4}\,ie(p'+p)^\mu \times \frac{i}{p^2-M^2+i0} \times ie(p+p')^\nu \times \frac{i}{p'^2-M^2+i0} \tag{S.49}$$

where $p' = p+k$. As usual, we combine the two denominators using Feynman parameter trick, thus

$$\frac{1}{p^2-M^2+i0} \times \frac{1}{(p+k)^2-M^2+i0} \;=\; \int\limits_0^1 dx\,\frac{1}{[q^2-\Delta+i0]^2} \tag{S.50}$$

where

$$q^2 - \Delta = x(p+k)^2 + (1-x)p^2 - M^2 \tag{S.51}$$

and hence

$$q = p + xk \quad \text{and} \quad \Delta = M^2 - x(1-x)k^2. \tag{S.52}$$

Next, we shift the integration variable from p to q, and this gives us

$$\Sigma^{\mu\nu}_{(1)}(k) = -ie^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{(p+p')^\mu (p+p')^\nu}{[q^2 - \Delta + i0]^2} \tag{S.53}$$

where in the numerator

$$\begin{aligned} (p+p')^\mu (p+p')^\nu &= (2q + (1-2x)k)^\mu (2q + (1-2x)k)^\nu \\ &= 4q^\mu q^\nu + (1-2x)^2 k^\mu k^\nu + 2(1-2x)[q^\mu k^\nu + k^\mu q^\nu] \end{aligned} \tag{S.54}$$

Note that the last term on the second line here is odd with respect to q and hence does not contribute to the $\int dq$ integral. As to the first term on the second line, in the context of $\int dq$ integral $q^\mu q^\nu$ is equivalent to $g^{\mu\nu} \times q^2/D$. Altogether, we have

$$\begin{aligned} (p+p')^\mu (p+p')^\nu &\cong \frac{4}{D} q^2 \times g^{\mu\nu} + (1-2x)^2 k^\mu k^\nu \\ &= \left(k^\mu k^\nu - k^2 g^{\mu\nu}\right) \times (1-2x)^2 + g^{\mu\nu} \times \left(\frac{4}{D} q^2 + (1-2x)^2 k^2\right), \end{aligned} \tag{S.55}$$

and consequently

$$\Sigma^{\mu\nu}_{(1)}(k) = \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) \times \Pi_{(1)}(k^2) + g^{\mu\nu} \times \Xi_{(1)}(k^2) \tag{S.56}$$

where

$$\Pi_{(1)}(k^2) = ie^2 \int_0^1 dx\, (1-2x)^2 \int_{\text{reg}} \frac{d^4q}{(2\pi)^4} \frac{1}{\left(q^2 - \Delta + i0\right)^2} \tag{S.57}$$

and

$$\Xi_{(1)}(k^2) = -ie^2 \int_0^1 dx \int_{\text{reg}} \frac{d^4q}{(2\pi)^4} \frac{\frac{4}{D} q^2 + (1-2x)^2 k^2}{\left(q^2 - \Delta + i0\right)^2}. \tag{S.58}$$

Our first task is to verify the tensor structure of the two-photon amplitude, so let us focus on the coefficient Ξ of the wrong tensor. Applying Wick rotation and dimensional regularization to the momentum integral in eq. (S.58), we calculate

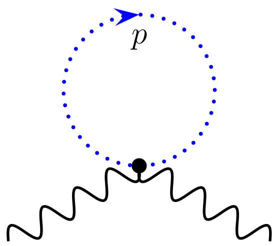
$$
\begin{aligned}
-i\int\limits_{\text{reg}}\frac{d^4q}{(2\pi)^4}\,\frac{\frac{4}{D}q^2\,+\,(1-2x)^2k^2}{\big(q^2-\Delta+i0\big)^2}\ &= \\
&= \int\limits_{\text{reg}}\frac{d^4q_E}{(2\pi)^4}\,\frac{-\frac{4}{D}q_E^2\,+\,(1-2x)^2k^2}{\big(q^2+\Delta\big)^2} \\
&= \mu^{4-D}\int\frac{d^Dq}{(2\pi)^D}\,\frac{-\frac{4}{D}q^2\,+\,(1-2x)^2k^2}{(q^2+\Delta)^2} \\
&= \mu^{4-D}\int\limits_0^\infty dt\,t\int\frac{d^Dq}{(2\pi)^D}\left(-\tfrac{4}{D}q^2\,+\,(1-2x)^2k^2\right)e^{-t(q^2+\Delta)} \\
&= \mu^{4-D}\int\limits_0^\infty dt\,te^{-t\Delta}\left(+\frac{4}{D}\frac{\partial}{\partial t}\,+\,(1-2x)^2k^2\right)\int\frac{d^Dq}{(2\pi)^D}\,e^{-tq^2} \\
&= \frac{\mu^{4-D}}{(4\pi)^{D/2}}\int\limits_0^\infty dt\,te^{-t\Delta}\left(\frac{4}{D}\frac{\partial}{\partial t}\,+\,(1-2x)^2k^2\right)t^{-D/2} \\
&= \frac{\mu^{4-D}}{(4\pi)^{D/2}}\int\limits_0^\infty dt\,te^{-t\Delta}\left(-2t^{-(D/2)-1}\,+\,(1-2x)^2k^2\times t^{-D/2}\right) \\
&= \frac{\mu^{4-D}}{(4\pi)^{D/2}}\left(-2\Gamma\left(1-\tfrac{D}{2}\right)\Delta^{\frac{D}{2}-1}\,+\,(1-2x)^2k^2\times\Gamma\left(2-\tfrac{D}{2}\right)\Delta^{\frac{D}{2}-2}\right) \\
&= \frac{\mu^{4-D}}{(4\pi)^{D/2}}\Gamma\left(1-\tfrac{D}{2}\right)\times\left[-2\Delta^{\frac{D}{2}-2}\,+\,\left(\tfrac{D}{2}-1\right)\Delta^{\frac{D}{2}-1}\times(1-2x)\,\frac{\partial\Delta}{\partial x}\right] \\
&= \frac{\mu^{4-D}}{(4\pi)^{D/2}}\,\Gamma\left(1-\tfrac{D}{2}\right)\times\frac{\partial}{\partial x}\left((1-2x)\Delta^{\frac{D}{2}-1}\right),
\end{aligned}
\tag{S.59}
$$

and consequently

$$
\begin{aligned}
\Xi_{(1)}(k^2) &= e^2 \int_0^1 dx \, \frac{\mu^{4-D}}{(4\pi)^{D/2}} \, \Gamma\left(1-\tfrac{D}{2}\right) \frac{\partial}{\partial x} \left((1-2x)\Delta^{\frac{D}{2}-1} \right) \\
&= \frac{e^2 \mu^{4-D}}{(4\pi)^{D/2}} \, \Gamma\left(1-\tfrac{D}{2}\right) \times \left[-\Delta^{\frac{D}{2}-1}\Big|_{x=1} - \Delta^{\frac{D}{2}-1}\Big|_{x=0} = -2(M^2)^{\frac{D}{2}-1} \right] \\
&= -\frac{\alpha M^2}{2\pi} \times \Gamma\left(1-\tfrac{D}{2}\right) \times \left(\frac{4\pi\mu^2}{M^2} \right)^{2-\frac{D}{2}} .
\end{aligned}
\tag{S.60}
$$

Note that thanks to $\Delta(x=1) = \Delta(x=0) = M^2$, the bottom line of eq. (S.60) is independent of the photon's momentum k. And since the second diagram's contribution $\Sigma^{\mu\nu}_{(2)}$ is also k–independent, this allows for the cancellation of the wrong tensor structure of the two-photon amplitude between the two diagrams.

Indeed, for the second diagram



(S.61)

we have

$$
i\Sigma^{\mu\nu}_{(2)} = \int \frac{d^4 p}{(2\pi)^4} \, 2ie^2 g^{\mu\nu} \times \frac{i}{p^2 - M^2 + i0} \tag{S.62}
$$

which does not depend on the photons' momenta and has wrong tensor structure

$$
\Sigma^{\mu\nu}_{(2)} = g^{\mu\nu} \times \Xi_{(2)} \, . \tag{S.63}
$$

To evaluate the coefficient $\Xi_{(2)}$ of this wrong tensor structure, we continue the loop momentum p

to Euclidean space and then use dimensional regularization, thus

$$
\begin{aligned}
\Xi_{(2)} &= \int\limits_{\text{reg}} \frac{d^4p}{(2\pi)^4} \frac{2ie^2}{p^2 - M^2 + i0} \\
&= \int\limits_{\text{reg}} \frac{d^4p_E}{(2\pi)^4} \frac{-2e^2}{-(p_E^2 + M^2)} \\
&= 2e^2\mu^{4-D} \int \frac{d^Dp_E}{(2\pi)^D} \int\limits_0^\infty dt\, e^{-t(M^2+p_E^2)} \\
&= 2e^2\mu^{4-D} \int\limits_0^\infty dt\, e^{-tM^2} \int \frac{d^Dp_E}{(2\pi)^D} e^{-tp_E^2} \\
&= \frac{2e^2\mu^{4-D}}{(4\pi)^{D/2}} \int\limits_0^\infty dt\, e^{-tM^2} t^{-D/2} \\
&= \frac{2e^2\mu^{4-D}}{(4\pi)^{D/2}} \times \Gamma\left(1 - \tfrac{D}{2}\right) (M^2)^{\frac{D}{2}-1} \\
&= \frac{\alpha M^2}{2\pi} \times \Gamma\left(1 - \tfrac{D}{2}\right) \times \left(\frac{4\pi\mu^2}{M^2}\right)^{2-\frac{D}{2}}
\end{aligned}
\tag{S.64}
$$

Comparing this formula to the wrong-tensor contribution (S.60) of the first diagram, we immediately see that they cancel each other,

$$
\Xi_{1\,\text{loop}} = \Xi_{(1)} + \Xi_{(2)} = 0 \tag{S.65}
$$

and therefore

$$
\Sigma^{\mu\nu}_{1\,\text{loop}} = \left(k^\mu k^\nu - k^2 g^{\mu\nu}\right) \times \Pi_{1\,\text{loop}}(k^2) \tag{1}
$$

$\mathcal{Q.E.D.}$

<u>Alternative solution</u>:

Above, we had to integrate over the Feynman parameter x to see that the wrong-tensor contribution from the two diagrams cancel each other. Alternatively, we may combine the two diagrams together before integrating over momenta or Feynman parameters. We can do that by identifying the loop momentum p of the second diagram with either p or $p' = p+k$ of the first diagram. For symmetry's sake, let's take the average of the two identifications, thus

$$\left[\frac{1}{p^2 - M^2 + i0}\right]_{(2)} \to \frac{1/2}{p^2 - M^2 + i0} + \frac{1/2}{p'^2 - M^2 + i0} = \frac{\frac{1}{2}p^2 + \frac{1}{2}p'^2 - M^2}{(p^2 - M^2 + i0)(p'^2 - M^2 + i0)} . \quad \text{(S.66)}$$

This trick brings both diagrams to a common denominator, so adding them up yields

$$i\Sigma^{\mu\nu}_{(1+2)} = e^2 \int \frac{d^4p}{(2\pi)^4} \frac{(p+p')^\mu (p+p')^\nu - g^{\mu\nu}(p^2 + p'^2 - 2M^2)}{(p^2 - M^2 + i0) \times (p'^2 - M^2 + i0)} . \quad \text{(S.67)}$$

At this point, we may simplify the common denominator using the Feynman's parameter trick. Proceeding exactly as in eqs. (S.50)–(S.52), we have

$$\Sigma^{\mu\nu}_{(1+2)} = -ie^2 \int_0^1 dx \int_{\text{reg}} \frac{d^4q}{(2\pi)^4} \frac{\mathcal{N}^{\mu\nu}}{[\ell^2 - \Delta + i0]^2} \quad \text{(S.68)}$$

where q and Δ are exactly as in eq. (S.52) and the numerator is

$$\begin{aligned} \mathcal{N}^{\mu\nu} &= (p+p')^\mu (p+p')^\nu - g^{\mu\nu}(p^2 + p'^2 - 2M^2) \\ &= (2q + (1-2x)k)^\mu (2q + (1-2x)k)^\mu - g^{\mu\nu}\Big((q - xk)^2 + (q + k - xk)^2 - 2M^2\Big) \\ &= 4q^\mu q^\nu + (1-2x)^2 k^\mu k^\nu + 2(1-2x)(q^\mu k^\nu + q^\nu k^\mu) \\ &\quad - g^{\mu\nu}\Big(2q^2 + (1-2x)k^2 - 2\Delta + 2(1-2x)(qk)\Big). \end{aligned} \quad \text{(S.69)}$$

In the context of the integral (S.68), we may ignore terms that are odd in q while $q^\mu q^\nu$ is equivalent to $q^2 \times g^{\mu\nu}/D$. Hence, the numerator (S.69) is equivalent to

$$\mathcal{N}^{\mu\nu} \cong (1-2x)^2 \times (k^\mu k^\nu - g^{\mu\nu}k^2) + g^{\mu\nu} \times \left(\frac{4}{D}q^2 - 2q^2 + 2\Delta\right) \quad \text{(S.70)}$$

and therefore

$$\Sigma^{\mu\nu}_{(1+2)}(k) = \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) \times \Pi_{(1+2)}(k^2) + g^{\mu\nu} \times \Xi_{(1+2)}(k^2) \quad \text{(S.71)}$$

where

$$\Pi_{(1+2)}(k^2) \;=\; ie^2 \int_0^1 dx\,(1-2x)^2 \int_{\text{reg}} \frac{d^4q}{(2\pi)^4}\,\frac{1}{\big(q^2-\Delta+i0\big)^2} \tag{S.72}$$

and

$$\Xi_{(1+2)}(k^2) \;=\; -ie^2 \int_0^1 dx \int_{\text{reg}} \frac{d^4q}{(2\pi)^4}\,\frac{2\Delta\;-\;(2-\frac{4}{D})q^2}{(q^2-\Delta+i0)^2}\,. \tag{S.73}$$

The wrong-tensor contribution (S.73) for the scalar loop looks exactly like the similar wrong-tensor contribution of the electron loop we have seen in class, and the momentum integral here vanishes in exactly the same way. Using dimensional regularization, we analytically continue to $\operatorname{Re} D < 2$ (because the integral diverges quadratically in $D=4$), and them we get zero for all D. Indeed,

$$\begin{aligned}
-i\int\frac{d^Dq}{(2\pi)^D}\,\frac{2\Delta\;-\;(2-\frac{4}{D})q^2}{(q^2-\Delta+i0)^2} \;&=\; \int\frac{d^Dq_E}{(2\pi)^D}\,\frac{2\Delta+(2-\frac{4}{D})q_E^2}{(q_E^2+\Delta)^2} \\
&= \int_0^\infty dt\,t\times e^{-\Delta t}\times\left(2\Delta\;-\;(2-\tfrac{4}{D})\frac{\partial}{\partial t}\right)\int\frac{d^Dq_E}{(2\pi)^D}\,e^{-tq_E^2} \\
&= \int_0^\infty dt\,t\times e^{-\Delta t}\times\left(2\Delta\;-\;(2-\tfrac{4}{D})\frac{\partial}{\partial t}\right)(4\pi t)^{-D/2} \\
&\propto \int_0^\infty dt\,e^{-\Delta t}\times\Big(2\Delta\times t^{1-(D/2)}\;+\;(D-2)\times t^{-D/2}\Big) \\
&= 2\Delta\times\Gamma(2-\tfrac{D}{2})\Delta^{2-(D/2)}\;+\;(D-2)\times\Gamma(1-\tfrac{D}{2})\Delta^{1-(D/2)} \\
&\qquad\langle\!\langle\text{using }\Gamma(2-\tfrac{D}{2})=\Gamma(1-\tfrac{D}{2})\times(1-\tfrac{D}{2})\rangle\!\rangle \\
&= \Delta^{1-(D/2)}\Gamma(1-\tfrac{D}{2})\times\Big[2(1-\tfrac{D}{2})\;+\;(D-2)\;=\;0\Big] \\
&= 0\;\forall D<2.
\end{aligned} \tag{S.74}$$

Analytically continuing this formula back to $D=4$ we have $\Xi_{1+2}(k^2)\equiv 0$ and hence $\Sigma^{\mu\nu}_{(1+2)}(k) = (g^{\mu\nu}k^2-k^\mu k^\nu)\times\Pi_{1+2}(k^2)$ as in eq. (1), *quod erat demonstrandum.*

Problem **2**(b):

Our next task is to calculate the $\Pi_{1\,\mathrm{loop}}(k^2)$ factor in eq. (1). As we saw in part (a), only the first diagram contributes to the correct tensor structure in $\Sigma^{\mu\nu}_{1\,\mathrm{loop}}$, hence according to eq. (S.57)

$$\Pi_{1\,\mathrm{loop}}(k^2) \;=\; ie^2\int_0^1 dx\,(1-2x)^2\int\limits_{\mathrm{reg}}\frac{d^4q}{(2\pi)^4}\,\frac{1}{\left(q^2-\Delta+i0\right)^2} \tag{S.75}$$

In the alternative solution, eq. (S.72) gives exactly the same formula for the $\Pi_{(1+2)}(k^2)$ for the net contribution of the two diagrams.

The momentum integral in eq. (S.75) should be rather familiar after so much related class-work and home-work, so let me simply write down the result:

$$i\int\limits_{\mathrm{reg}}\frac{d^4q}{(2\pi)^4}\,\frac{1}{\left(q^2-\Delta+i0\right)^2} \;=\; \frac{-1}{16\pi^2}\,\Gamma\left(2-\tfrac{D}{2}\right)\left(\frac{4\pi\mu^2}{\Delta}\right)^{2-\frac{D}{2}} \;\xrightarrow[D\to4]{}\; \frac{-1}{16\pi^2}\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{\Delta}\right) \tag{S.76}$$

where

$$\frac{1}{\bar\epsilon} \;=\; \frac{2}{4-D}\;-\;\gamma_E\;+\;\log(4\pi)\,. \tag{S.8}$$

Consequently,

$$\begin{aligned}\Pi_{1\,\mathrm{loop}}(k^2) \;&=\; -\frac{\alpha}{4\pi}\int_0^1 dx\,(1-2x)^2\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{M^2-x(1-x)k^2}\right)\\ &=\; -\frac{\alpha}{12\pi}\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{M^2}\;+\;\hat I(k^2/M^2)\right)\end{aligned} \tag{S.77}$$

where

$$\hat I(k^2/M^2) \;=\; 3\int_0^1 dx\,(1-2x)^2\log\frac{M^2}{M^2-x(1-x)k^2} \;=\; 3I(k^2/M^2)\;-\;2J(k^2/M^2). \tag{S.78}$$

Finally, note that eq. (S.77) is the bare one-loop amplitude, without accounting for the counterterms. Adding the counterterm δ_3 to the picture, we have

$$\Sigma^{\mu\nu}_{\rm net}(k) \;=\; \Sigma^{\mu\nu}_{\rm loops}(k) \;-\; \delta_3(k^2 g^{\mu\nu} - k^\mu k^\nu) \tag{S.79}$$

and hence

$$\Pi_{\rm net}(k^2) \;=\; \Pi_{\rm loops}(k^2) \;-\; \delta_3\,. \tag{S.80}$$

The renormalization condition for the δ_3 is $\Pi_{\rm net} = 0$ for $k^2 = 0$; this assures that the dressed propagator for the renormalized EM field has a pole at $k^2 = 0$ with residue $= 1$. Consequently

$$\delta_3 \;=\; \Pi_{\rm loops}(k^2 = 0), \tag{S.81}$$

hence given the one-loop formula (S.77) where $\hat{I}(0) = 0$ (*cf.* eq. (S.78)), the counterterm is

$$\Delta_3 \;=\; -\frac{\alpha}{12\pi}\left(\frac{1}{\bar{\epsilon}} + \log\frac{\mu^2}{M^2}\right) +\; O(\alpha^2/\pi^2), \tag{S.82}$$

and the net propagator correction is

$$\Pi_{\rm net}(k^2) \;=\; -\frac{\alpha}{12\pi}\,\hat{I}(k^2/M^2) \;+\; O(\alpha^2/\pi^2). \tag{S.83}$$

Problem **2**(c):

At high momenta $k^2 \gg M^2$, we may approximate

$$\log\frac{M^2}{M^2 - x(1-x)k^2} \;\approx\; \log\frac{M^2}{-x(1-x)k^2} \;+\; O(M^2/k^2) \tag{S.84}$$

and hence

$$\hat{I}(k^2/M^2) \;\approx\; 3\int_0^1 dx\,(1-2x)^2\left[-\log\frac{-k^2}{M^2} \;+\; \log\frac{1}{x(1-x)}\right] \;=\; -\log\frac{-k^2}{M^2} \;+\; \frac{8}{3}\,. \tag{S.85}$$

Consequently, at high momenta the “vacuum polarization” factor $\Pi(k^2)$ behaves as

$$\Pi(k^2) \;=\; \frac{\alpha}{12\pi}\left(+\log\frac{-k^2}{M^2} \;-\; \frac{8}{3} \;+\; O(M^2/k^2)\right) +\; O(\alpha^2), \tag{S.86}$$

and therefore the effective gauge coupling

$$\alpha_{\text{eff}}(k^2) \;=\; \frac{\alpha}{1-\Pi(k^2)} \tag{S.87}$$

behaves according to

$$\frac{1}{\alpha_{\text{eff}}(k^2)} \;\approx\; \frac{1}{\alpha(0)} \;-\; \frac{1}{12\pi}\left(\log\frac{-k^2}{M^2} \;-\; \frac{8}{3}\right). \tag{2}$$

Quod erat demonstrandum.