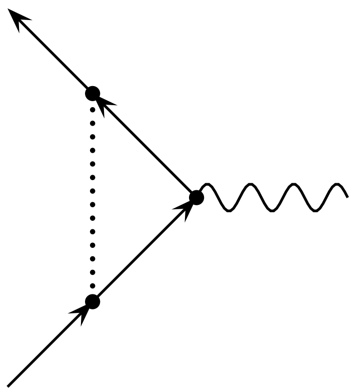
Problem **2**(a):

The scalar field affects the muon's QED vertex through loop diagrams containing the scalar's propagators. At the one-loop level, there is one such diagram



(S.1)

which contributes

$$\begin{aligned}\Delta_S[ie\Gamma^\mu(p',p)] &= \int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-M_s^2+i0}\times(-ig)\frac{i}{\not p'+\not k-m+i0}(ie\gamma^\mu)\frac{i}{\not p+\not k-m+i0}(-ig)\\ &= -eg^2\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{\mathcal{D}}\end{aligned} \tag{S.2}$$

where the numerator is

$$\mathcal{N}^\mu = (\not p'+\not k+m)\,\gamma^\mu\,(\not p+\not k+m) \tag{S.3}$$

and the denominator is

$$\begin{aligned}\frac{1}{\mathcal{D}} &= \frac{1}{k^2-M_s^2+i0}\times\frac{1}{(k+p')^2-m^2+i0}\times\frac{1}{(k+p)^2-m^2+i0}\\ &= \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{(\ell^2-\Delta+i0)^3}\,.\end{aligned} \tag{S.4}$$

As usual, in the Feynman parameter integral

$$\ell^2 \;-\; \Delta \;=\; z(k^2 - M_s^2) \;+\; x((k+p)^2 - m^2) \;+\; y((k+p')^2 - m^2) \tag{S.5}$$

and hence

$$\ell \;=\; k \;+\; xp \;+\; yp', \tag{S.6}$$
$$\Delta \;=\; zM_s^2 \;+\; (1-z)^2 m^2 \;-\; xyq^2 \tag{S.7}$$

for the on-shell muon momenta, $p^2 = p'^2 = m^2$.

Altogether, we now have

$$\Delta_S\Gamma^\mu(p',p) \;=\; 2ig^2 \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{(\ell^2-\Delta+i0)^3}\,, \tag{S.8}$$

and our next task is to simplify the numerator (S.3) in the present context. That is, we re-express $\mathcal{N}^\mu$ in terms of the shifted loop momentum ℓ, discard terms which integrate to zero (because they are odd with respect to $\ell \to -\ell$ or $x \leftrightarrow y$ symmetries), and also make use of the $\bar{u}(p')\,\Gamma^\mu\,u(p)$ context which allows us to substitute $\not{p} \to m$ in the rightmost factor and $\not{p}' \to m$ in the leftmost factor. Thus, proceeding similarly to QED vertex correction (*cf.* the supplementary notes), we obtain

$$\begin{aligned}
\mathcal{N}^\mu \;&=\; \big((\not{\ell} - x\not{p} - y\not{p}') + \not{p}' + m\big)\,\gamma^\mu\,\big((\not{\ell} - x\not{p} - y\not{p}') + \not{p} + m\big) \\
&\cong \not{\ell}\gamma^\mu\not{\ell} \;+\; (z\not{p}' + x\not{q} + m)\,\gamma^\mu\,(z\not{p} - y\not{q} + m) \\
&\cong\; \gamma^\lambda\gamma^\mu\gamma^\nu \times \frac{\ell^2 g_{\lambda\nu}}{D} \;+\; ((z+1)m + x\not{q})\,\gamma^\mu\,((z+1)m - y\not{q}) \\
&=\; \frac{2-D}{D}\,\ell^2\,\gamma^\mu \;+\; (z+1)^2 m^2\,\gamma^\mu \;-\; xy\not{q}\gamma^\mu\not{q} \;+\; (z+1)m\Big((x-y)q^\mu \;+\; (x+y)i\sigma^{\mu\nu}q_\nu\Big) \\
&\cong\; \mathcal{N}_1 \times \gamma^\mu \;+\; \mathcal{N}_2 \times \frac{i\sigma^{\mu\nu}q_\nu}{2m}
\end{aligned} \tag{S.9}$$

where

$$\mathcal{N}_1 \;=\; -\frac{D-2}{D}\,\ell^2 \;+\; (1+z)^2 m^2 \;+\; xyq^2 \tag{S.10}$$

$$\text{and} \quad \mathcal{N}_2 \;=\; 2(1-z^2)m^2. \tag{S.11}$$

In light of the Dirac-matrix structure of the last line of eq. (S.9), the $\mathcal{N}_1$ contributes to the F_1 form-factor of the muon while the $\mathcal{N}_2$ contributes to the F_2 form factor,

$$\begin{aligned} \Delta_S F_1(q^2) \;&=\; 2ig^2 \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int\limits_{\text{reg}} \frac{d^4\ell}{(2\pi)^4} \frac{\mathcal{N}_1}{(\ell^2-\Delta+i0)^3} \\ &\quad - \text{ similar integral for } q^2=0 \text{ because of } \Delta_S\delta_1\,, \end{aligned} \tag{S.12}$$

$$\Delta_S F_2(q^2) \;=\; 2ig^2 \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int\limits_{\text{reg}} \frac{d^4\ell}{(2\pi)^4} \frac{\mathcal{N}_2}{(\ell^2-\Delta+i0)^3}\,. \tag{S.13}$$

In this exercise, we are interested in the anomalous magnetic moment of the muon, so all we need is F_2 for $q^2=0$, and we do not need to worry about the counterterm δ_1 because it affects only the other form factor F_1. In eq. (S.13)for the F_2, the numerator $\mathcal{N}_2$ does not depend on the loop momentum ℓ, so the $\int d^4\ell$ integral converges without any regularization, UV or IR,

$$\begin{aligned} i\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2-\Delta+i0)^3} \;&=\; \int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{(\ell_E^2+\Delta)^3} \\ &=\; \frac{1}{16\pi^2} \int\limits_0^2 d\ell_E^2 \frac{\ell_E^2}{(\ell_E^2+\Delta)^3} \\ &=\; \frac{1}{16\pi^2} \times \frac{1}{2\Delta}\,. \end{aligned} \tag{S.14}$$

Consequently,

$$\Delta_S F_2(q^2) \;=\; \frac{g^2}{16\pi^2} \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\, \frac{2m^2(1-z^2)}{\Delta} \tag{S.15}$$

where Δ is as in eq. (S.7), and hence for $q^2 = 0$

$$\begin{aligned}
\Delta_S\left(\frac{g_\mu - 2}{2}\right) &= \Delta_S F_2(q^2 = 0) \\
&= \frac{g^2}{16\pi^2} \iiint_0^1 dx\, dy\, dz\, \delta(x+y+z-1)\, \frac{2m^2(1-z^2)}{zM_s^2 + (1-z)^2 m^2} \\
&= \frac{g^2}{16\pi^2} \int_0^1 dz\, (1-z) \times \frac{2m^2(1-z^2)}{zM_s^2 + (1-z)^2 m^2}\,.
\end{aligned} \tag{S.16}$$

The last integral here is a complicated function of the muon-to-scalar mass ratio m/M_s, but for the problem at hand, the scalar is much heavier than the muon. Hence, we approximate the denominator according to

$$\begin{aligned}
zM_s^2 \;+\; (1-z)^2 m^2 \;&\approx \begin{cases} zM_s^2 \;+\; 0 & \text{except when } z \approx 0 \\ zM_s^2 \;+\; m^2 & \text{for } z \approx 0 \end{cases} \\
&\approx\; zM_s^2 \;+\; m^2 \quad \text{for all } z,
\end{aligned} \tag{S.17}$$

and consequently evaluate

$$\int_0^1 dz\, \frac{2m^2(1-z^2)(1-z)}{zM_s^2 + m^2} \;=\; 2\,\frac{m^2}{M_s^2}\left(\log\frac{M_s^2}{m^2} \;-\; \frac{7}{6} \;+\; O\left(\frac{m^2}{M^2}\right)\right). \tag{S.18}$$

Thus, to the leading orders in the Yukawa coupling g and in the m/M_s mass ration, the scalar's effect on the anomalous magnetic moment of the muon amounts to

$$\Delta_S g_\mu \;\approx\; \frac{g^2}{4\pi^2}\,\frac{m^2}{M_s^2}\left(\log\frac{M_s^2}{m^2} \;-\; \frac{7}{6}\right). \tag{S.19}$$

Experimentally, muon's anomalous magnetic moment agrees with the MSM (Minimal Standard Model) to 8 significant digits; beyond that, we have eqs. (1) and (2). Interpreting these equations as limits on contributions from outside the MSM — *i.e.*, as limit on the $\Delta_S g_\mu$, we have

$$\Delta_S g_\mu \;<\; 93\cdot 10^{-10} \tag{S.20}$$

at 95% confidence level.[⋆] In light of eq. (S.19), this limit amounts to a limit on the Yukawa coupling of the scalar Φ to the muon field,

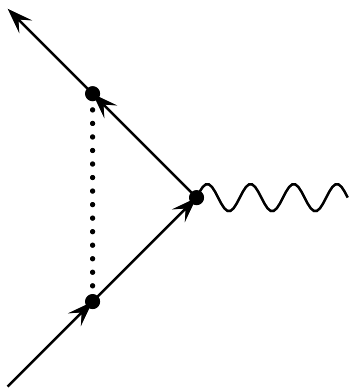
$$g \;<\; 0.3 \quad \text{for } M_S = 200\,\text{GeV} \tag{S.21}$$

or more generally

$$g \;<\; 0.3\times\left(\frac{M_S}{200\,\text{GeV}}\right). \tag{S.22}$$

Problem **2**(b):

At the one-loop level, the axion's effect on the QED vertex of the muon follows from a single diagram



(S.23)

which looks exactly like (S.1) but evaluates differently because of different Yukawa vertices:

$$\frac{2m_\mu}{f_a}\gamma^5 \;\equiv\; g\gamma^5 \quad \text{instead of } -ig. \tag{S.24}$$

Also, the axion is lighter than the muon, $M_a \ll m_\mu$.

⋆ The RHS here is the central value from eq. (1) plus two sigmas, statistical and systematic errors being added in quadrature. The central value is taken from eq. (1) rather than eq. (2) because it allows for a bigger effect of the scalar Φ.

Consequently,

$$\Delta_a[ie\Gamma^\mu(p',p)] = \int\limits_{\rm reg}\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-M_a^2+i0}\times(g\gamma^5)\frac{i}{\not p'+\not k-m+i0}(ie\gamma^\mu)\frac{i}{\not p+\not k-m+i0}(g\gamma^5)$$
$$= -2eg^2\iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\limits_{\rm reg}\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{(\ell^2-\Delta+i0)^3} \tag{S.25}$$

where the denominator is exactly as in part (a) of the problems (*cf.* eqs. (S.6) and (S.7)) except for $M_s^2\to M_a^2$, but the numerator is now

$$\begin{aligned}\mathcal{N}^\mu &= -\gamma^5\times(\not k+\not p'+m)\times\gamma^\mu\times(\not k+\not p+m)\times\gamma^5\\ &= +(\not k+\not p'-m)\times\gamma^\mu\times(\not k+\not p-m).\end{aligned} \tag{S.26}$$

As in part (a), we need to re-express this numerator in terms of the shifted loop momentum ℓ and then discard terms which integrate to zero or vanish on-shell (in the context of $\bar u'\Gamma^\mu u$). Proceeding similarly to eq. (S.9), we obtain

$$\mathcal{N}^\mu \cong \mathcal{N}_1\times\gamma^\mu + \mathcal{N}_2\times\frac{i\sigma^{\mu\nu}q_\nu}{2m} \tag{S.27}$$
$$\text{where}\quad \mathcal{N}_1 = -\frac{D-2}{D}\,\ell^2 + (1-z)^2m^2 + xyq^2 \tag{S.28}$$
$$\text{and}\quad \mathcal{N}_2 = -2(1-z)^2m^2. \tag{S.29}$$

Again, the $\mathcal{N}_1$ affects the F_1 form factor of the muon while the $\mathcal{N}_2$ affects the F_2 form factor. The anomalous magnetic moment follows from the latter, which is given by

$$\begin{aligned}\Delta_aF_2(q^2) &= 2ig^2\iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_2}{(\ell^2-\Delta+i0)^3}\\ &= \frac{g^2}{16\pi^2}\iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\times\frac{\mathcal{N}_2}{\Delta}\,,\end{aligned} \tag{S.30}$$

where the momentum integral is exactly as in eq. (S.14) because the numerator $\mathcal{N}_2$ does not depend on the loop momentum ℓ. It also depends on only one Feynman parameter — z but not x or y — and for $q^2 = 0$ so does the denominator $\Delta \to M_a^2 + (1-z)^2 m^2$. Therefore,

$$\begin{aligned} \Delta_a\left(\frac{g_\mu - 2}{2}\right) &= \Delta_a F_2(q^2 = 0) \\ &= \frac{g^2}{16\pi^2} \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{-2m^2(1-z)^2}{zM_a^2 + (1-z)^2 m^2} \\ &= \frac{g^2}{16\pi^2} \int_0^1 dz\,(1-z) \times \frac{-2m^2(1-z)^2}{zM_a^2 + (1-z)^2 m^2}\,. \end{aligned} \tag{S.31}$$

Unlike the scalar field in part (a) of the problem, the axion is light compared to the muon, so the approximation (S.17) does not apply here. Instead, for $M_a \ll m_\mu$ we simply neglect the axion's mass in the denominator of eq. (S.31),

$$\int_0^1 dz\,(1-z) \times \frac{-2m^2(1-z)^2}{zM_a^2 + (1-z)^2 m^2} \approx \int_0^1 dz\,(1-z) \times \frac{-2m^2(1-z)^2}{(1-z)^2 m^2} = -1 \tag{S.32}$$

and hence

$$\Delta_a g_\mu \approx -\frac{g^2}{8\pi^2} = -\frac{m_\mu^2}{2\pi^2 f_a^2}\,. \tag{S.33}$$

Because the negative sign of the axion's effect on the muon's magnetic moment is opposite from the sign of discrepancy (1) between the experiment and the Minimal Standard Model, a theory made out of MSM plus an axion plus nothing else seems to be ruled out. However, if we use the alternative method for calculating the hadronic loops in MSM which leads to eq. (2) instead of eq. (1), then we have a little room for a negative contribution of the axion as long as

$$\Delta_a g_\mu > -20 \cdot 10^{-10}. \tag{S.34}$$

In light of eq. (S.33), this leads to an upper limit on the axion's Yukawa coupling and hence to a lower limit on the axion's scale,

$$g < 2 \cdot 10^{-4} \quad\Longrightarrow\quad f_a > 10^4\, m_\mu \approx 1000\,\text{GeV}. \tag{S.35}$$

Problem **3**:

The δ_2 and δ_m counterterms of QED are related to the electron's self-energy correction

$$\Sigma^{\rm tot}(\not{p}) \;=\; \Sigma^{\rm loops}(\not{p}) \;+\; \delta_m \;-\; \delta_2 \times \not{p} \tag{S.36}$$

which satisfies renormalization conditions

$$\textbf{for} \quad \not{p} = m, \quad \textbf{both} \quad \Sigma^{\rm tot} = 0 \quad \textbf{and} \quad \frac{d\Sigma^{\rm tot}}{d\not{p}} = 0. \tag{S.37}$$

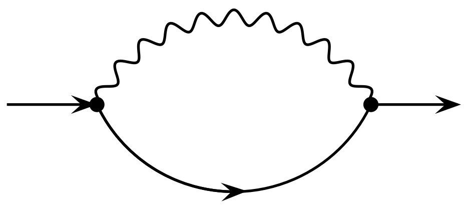
Thanks to these conditions,

$$\delta_2 \;=\; \frac{d\Sigma^{\rm loops}}{d\not{p}}\bigg|_{\not{p}=m} \tag{S.38}$$

and also

$$\delta_m \;=\; m \times \delta_2 \;-\; \Sigma^{\rm loops}\Big|_{\not{p}=m}. \tag{S.39}$$

At the one loop level of analysis, there is only one diagram



(S.40)

for the electron's self energy, which yields

$$-i\Sigma^{1\,\rm loop}(\not{p}) \;=\; \int\limits_{\rm reg} \frac{d^4k}{(2\pi)^4}\,\frac{-ig^{\lambda\nu}}{k^2+i0} \times ie\gamma_\lambda\,\frac{i}{\not{k}+\not{p}-m_e+i0} \times ie\gamma_\nu\,. \tag{S.41}$$

For consistency with the calculation of the δ_1 counterterm in the notes I distributed in class, we need to use the same regulators here: a tiny photon's mass m_γ to regulate the infrared divergence, and dimension $D < 4$ to regulate the UV divergence. Thus,

$$\begin{aligned}\Sigma^{1\,\text{loop}}(\not p) &= i\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\,\frac{-ig^{\lambda\nu}}{k^2-m_\gamma^2+i0}\times ie\gamma_\lambda\,\frac{i}{\not k+\not p-m_e+i0}\times ie\gamma_\nu\\ &= -ie^2\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\,\frac{\mathcal{N}}{\mathcal{D}}\end{aligned} \tag{S.42}$$

where the numerator is

$$\mathcal{N} \;=\; \gamma^\nu(\not k+\not p+m_e)\gamma_\nu \;=\; Dm_e \;-\; (D-2)(\not p+\not k) \tag{S.43}$$

and the denominator is

$$\frac{1}{\mathcal{D}} \;=\; \frac{1}{k^2-m_\gamma^2+i0}\times\frac{1}{(k+p)^2-m_e^2+i0} \;=\int\limits_0^1 dx\,\frac{1}{(\ell^2-\Delta+i0)^2} \tag{S.44}$$

for

$$\ell^2 \;-\; \Delta \;=\; (1-x)(k^2-m_\gamma^2) \;+\; x((k+p)^2-m_e^2) \tag{S.45}$$

and hence

$$\ell \;=\; k \;+\; xp, \tag{S.46}$$

$$\Delta \;=\; xm_e^2 \;-\; x(1-x)p^2 \;+\; (1-x)m_\gamma^2\,. \tag{S.47}$$

As usual, we re-express the numerator (S.43) in terms of the shifted loop momentum ℓ and then discard odd powers of ℓ, thus

$$\mathcal{N} \;=\; Dm_e \;-\; (D-2)(\not\ell - x\not p+\not p) \;\cong\; Dm_e \;-\; (D-2)(1-x)\,\not p, \tag{S.48}$$

and therefore

$$\Sigma^{1\,\text{loop}}(\not p) \;=\; e^2\int\limits_0^1 dx\,\Big(Dm_e \;-\; (D-2)(1-x)\,\not p\Big)\times\int\limits_{\text{reg}}\frac{d^4\ell}{(2\pi)^4}\,\frac{-i\mu^{4-D}}{(\ell^2-\Delta+i0)^2}\,. \tag{S.49}$$

The momentum integral here should be familiar to you by now, so let me simply write it down,

$$\frac{d^4\ell}{(2\pi)^4}\,\frac{-i\mu^{4-D}}{(\ell^2-\Delta+i0)^2}\;=\;\frac{\mu^{4-D}}{(4\pi)^{D/2}}\,\Gamma\left(2-\tfrac{D}{2}\right)\Delta^{\frac{D}{2}-2}\;=\;\frac{1}{16\pi^2}\,\Gamma(\epsilon)\,\left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon}. \tag{S.50}$$

Consequently,

$$\Sigma^{1\,\text{loop}}(\not p)\;=\;\frac{\alpha}{2\pi}\,\Gamma(\epsilon)(4\pi\mu^2)^{\epsilon}\int_0^1\!dx\,\frac{(2-\epsilon)m_e\,-\,(1-\epsilon)(1-x)\not p}{\Delta^{\epsilon}(z)}\,. \tag{S.51}$$

In the absence of the IR regulator (*i.e.*, for $m_\gamma^2=0$), the integral (S.51) converges for $\epsilon<1$ (*i.e.*, for $D>2$) and off-shell momenta $p^2<m_e^2$. Consequently, $\Sigma(\not p)$ can be analytically continued to any complex D and $\not p$ and the IR regulator seems un-necessary. Unfortunately, this continuation has a mild (for small ϵ) singularity for $p^2=m_e^2$: The $\Sigma(\not p)$ is continuous but the derivative $d\Sigma/d\not p$ becomes infinite. Indeed, taking the derivative of eq. (S.51) with respect to $\not p$, we have

$$\frac{d\Sigma^{1\,\text{loop}}}{d\not p}\;=\;\frac{\alpha}{2\pi}\,\Gamma(\epsilon)(4\pi\mu^2)^{\epsilon}\int_0^1\!dx\left(\begin{aligned}&\frac{-(1-\epsilon)(1-x)}{\Delta^{\epsilon}}\;-\;\epsilon\frac{(2-\epsilon)m_e-(1-\epsilon)(1-x)\not p}{\Delta^{1+\epsilon}}\times\\&\qquad\qquad\times\left[\frac{\partial\Delta}{\partial\not p}\;=\;-2x(1-x)\not p\right]\end{aligned}\right). \tag{S.52}$$

Let us neglect the IR regulator for a moment and take $m_\gamma^2=0$ so $\Delta=x\times(m^2-p^2+xp^2)$. Then for $x\to0$, the integrand of eq. (S.52) behaves as

$$\frac{1}{x^{\epsilon}}\times\left[-\frac{1-\epsilon}{[m^2-p^2+xp^2]^{\epsilon}}\;+\;\frac{2\epsilon\not p((2-\epsilon)m-(1-\epsilon)\not p)}{[m^2-p^2+xp^2]^{1+\epsilon}}\right]. \tag{S.53}$$

For off-shell momenta $p^2<m^2$, the expression in the square brackets here is finite and the $\int dx\,x^{-\epsilon}$ is perfectly finite as long as $\epsilon<1$ *i.e.*, $D>2$. But for the on-shell momentum $p^2=m^2$, the second term in the square brackets blows up at $x\to0$, we end up with

$$\int_0^1\!dx\,\frac{\text{finite}}{x^{1+2\epsilon}}\,, \tag{S.54}$$

which diverges for any $\epsilon\geq0$ *i.e.*, $D\leq4$. And that's why we need the IR regulator $m_\gamma^2>0$.

So let us put the IR regulator back where its belongs and calculate the derivative (S.52) for the on-shell momentum. For $\not p = m_e$, the integrand on the RHS of eq. (S.52) simplifies to

$$-\frac{(1-\epsilon)(1-x)}{\Delta^\epsilon} + \frac{2\epsilon x(1-x)[1+(1-\epsilon)x]\times m_e^2}{\Delta^{1+\epsilon}} \tag{S.55}$$

where

$$\Delta = x^2 m_e^2 + (1-x)m_\gamma^2 \approx x^2 m_e^2 + m_\gamma^2 . \tag{S.56}$$

The approximation here follows from the IR regulator being important only for $z \to 0$, and it allows us to extract a total derivative from the integrand:

$$\text{(S.55)} = -\frac{d}{dx}\left(\frac{(1-x)[1+(1-\epsilon)x]}{\Delta^\epsilon}\right) - \frac{1+(1-\epsilon)x}{\Delta^\epsilon} . \tag{S.57}$$

For $\epsilon < \frac{1}{2}$ — *i.e.*, for $D > 3$ — the second term here can be integrated without the photon's mass,

$$\int\limits_0^1 dx \, \frac{1+(1-\epsilon)x}{[\Delta = x^2 m_e^2]^\epsilon} = \frac{1}{m_e^{2\epsilon}} \times \left(\frac{1}{1-2\epsilon} + \frac{1-\epsilon}{2-2\epsilon}\right), \tag{S.58}$$

hence

$$\int\limits_0^1 dx \, \text{(S.55)} = \frac{1}{m_\gamma^{2\epsilon}} - \frac{1}{m_e^{2\epsilon}} \times \left(\frac{1}{1-2\epsilon} + \frac{1}{2}\right) \tag{S.59}$$

and therefore

$$\begin{aligned} \delta_2^{\text{order}\,\alpha} &= \left.\frac{d\Sigma^{1\,\text{loop}}}{d\not p}\right|_{\not p = m} \\ &= \frac{\alpha}{2\pi}\,\Gamma(\epsilon) \left(\frac{4\pi\mu^2}{m_e^2}\right)^\epsilon \times \left[\left(\frac{m_e^2}{m_\gamma^2}\right)^\epsilon - \frac{1}{1-2\epsilon} - \frac{1}{2}\right] \\ &= -\frac{\alpha}{4\pi}\,\Gamma(\epsilon) \left(\frac{4\pi\mu^2}{m_e^2}\right)^\epsilon \times \left[1 + \frac{4\epsilon}{1-2\epsilon} + 2 - 2\left(\frac{4\pi\mu^2}{m_e^2}\right)\right] \end{aligned} \tag{S.60}$$

In the $D \to 4$ limit, this counterterm becomes

$$\delta_2^{\text{order}\,\alpha} = -\frac{\alpha}{4\pi}\left[\frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{m_e^2} + 4 - 2\log\frac{m_e^2}{m_\gamma^2}\right]. \tag{S.61}$$

By comparison, in the notes distributed in class, I have calculated

$$\begin{aligned}\delta_1^{\text{order}\,\alpha} &= -\frac{\alpha}{4\pi}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^{\epsilon} \times \left[1 + \frac{4\epsilon}{1-2\epsilon} + 2 - 2\left(\frac{4\pi\mu^2}{m_e^2}\right)\right] \\ &\xrightarrow[D\to 4]{} -\frac{\alpha}{4\pi}\left[\frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{m_e^2} + 4 - 2\log\frac{m_e^2}{m_\gamma^2}\right].\end{aligned} \tag{S.62}$$

Thus, $\delta_2 = \delta_1$ (to order α), and this equality holds for any dimension between 3 and 4. *Quod erat demonstrandum.*

In fact, the identity $\delta_1 = \delta_2$ holds in any spacetime dimension and even for a finite photon mass $m_\gamma \not\ll m_e$, but proving *that* goes beyond the scope of this exercise.