Problem **1**(a):

As explained in class, at high momenta $p^2 \gg m^2$ we may approximate the electron's propagator as

$$\frac{i}{\not{p} - m + i0} = \frac{i(\not{p} + m)}{p^2 - m^2 + i0} \approx \frac{i(\not{p} + m)}{p^2 + i0} \tag{S.1}$$

The m^2 term in the denominator becomes negligible at high energies, but the m term in the numerator remains important (for some processes) because it changes the electron helicity (in the context of propagator $\times$ vertex). In other words, at high energies m acts as a valence $= 2$ coupling between the left and right chiralities of the electron, but its role as a mass is not important. Consequently, $m(E)$ renormalizes like all the other couplings a QFT.

Specifically, the renormalized mass $m(E)$ is related to the bare mass m_b according to

$$m_b = \frac{m(E) + \delta_m(E)}{1 + \delta_2(E)}. \tag{S.2}$$

Since the bare mass does not depend on the renormalization point, the renormalized mass and the counterterms satisfy

$$\frac{dm}{d\log E} = (m + \delta_m) \times \frac{d\delta_2}{d\log E} - \frac{d\delta m}{d\log E} = (m + \delta_m) \times 2\gamma_e - \frac{d\delta m}{d\log E}. \tag{S.3}$$

At the one-loop level this formula simplifies to

$$\frac{dm}{d\log E} = 2m\gamma_2 - \frac{d\delta m}{d\log E}. \tag{S.4}$$

In QED, the δ_m counterterm is proportional to the electron's mass itself,

$$\delta m(E) = m \times \hat{\delta}(E), \tag{S.5}$$

because for $m = 0$ the theory has a chiral symmetry which leads to $\delta_m = 0$. Plugging eq. (S.5) into eq. (S.3) we get

$$\frac{dm}{d\log E} \;=\; 2m(1+\hat{\delta})\gamma_e \;-\; m\,\frac{d\hat{\delta}}{d\log E}\,, \tag{S.6}$$

or equivalently

$$\frac{dm}{d\log E} \;=\; m\times\gamma_m(\alpha(E)) \tag{1}$$

where

$$\gamma_m \;=\; 2\gamma_e\times(1+\hat{\delta}) \;-\; \frac{d\hat{\delta}}{d\log E}\,. \tag{S.7}$$

In the Minimal Subtraction regularization scheme the counterterms generally look like

$$\begin{aligned}
\delta_1(\epsilon,\alpha) \;=\; \delta_2(\epsilon,\alpha) &\;=\; \frac{C_2(\alpha)}{\epsilon} \;+\; \text{higher poles},\\
\delta_3(\epsilon,\alpha) &\;=\; \frac{C_3(\alpha)}{\epsilon} \;+\; \text{higher poles},\\
\hat{\delta}(\epsilon,\alpha) &\;=\; \frac{\hat{C}(\alpha)}{\epsilon} \;+\; \text{higher poles},
\end{aligned} \tag{S.8}$$

In terms of such counterterms, the anomalous dimension of the electron field Ψ is

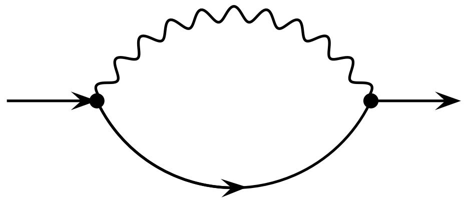
$$\gamma_e(\alpha) \;=\; -\alpha\frac{d}{d\alpha}C_2(\alpha) \tag{S.9}$$

while the anomalous dimension (S.7)of the electron's mass becomes

$$\gamma_m(\alpha) \;=\; \alpha\frac{d}{\alpha}(\hat{C}-C_2). \tag{S.10}$$

Problem **1**(b):

The δ_2 and δ_m counterterms of QED cancel the divergences of the electron self-energy correction $\Sigma(\not{p})$. At the one-loop level, the self-energy correction comes from a single diagram



(S.11)

which yields

$$-i\Sigma^{1\,\text{loop}}(\not{p}) \;=\; \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\, ie\gamma_\mu \frac{i}{\not{k}+\not{p}-m_e+i0} \times ie\gamma_\nu \times \frac{-i}{k^2+i0}\left(g^{\mu\nu} \;+\; (\xi-1)\,\frac{k^\mu k^\nu}{k^2}\right). \quad \text{(S.12)}$$

Note that we do not fix the Feynman gauge here but allow for a general gauge parameter ξ for the photon propagator (2).

For large loop momentum $k \gg p, m$ we may expand the fermion propagator in powers of $(m-\not{p})/\not{k}$,

$$\frac{1}{\not{k}+\not{p}-m+i0} \;=\; \frac{1}{\not{k}+i0} + \frac{1}{\not{k}+i0}(m-\not{p})\frac{1}{\not{k}+i0} + \frac{1}{\not{k}+i0}(m-\not{p})\frac{1}{\not{k}+i0}(m-\not{p})\frac{1}{\not{k}+i0} + \cdots. \quad \text{(S.13)}$$

Only the first two terms in this expansion contribute to the UV divergence of the integral (S.12), thus

$$\Sigma^{1\,\text{loop}}_{\text{div}}(\not{p}) \;=\; -ie^2 \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i0}\left(g^{\mu\nu} \;+\; (\xi-1)\,\frac{k^\mu k^\nu}{k^2}\right) \times$$
$$\times\, \gamma_\mu\left(\frac{1}{\not{k}+i0} + \frac{1}{\not{k}+i0}(m-\not{p})\frac{1}{\not{k}+i0}\right)\gamma_\nu\,, \quad \text{(S.14)}$$

$$\Sigma^{1\,\text{loop}}(\not{p}) \;=\; \Sigma^{1\,\text{loop}}_{\text{div}}(\not{p}) \;+\; \text{finite}(p).$$

On the second line here, we have

$$\gamma_\mu\left(\frac{1}{\not{k}+i0} + \frac{1}{\not{k}+i0}(m-\not{p})\frac{1}{\not{k}+i0}\right)\gamma_\nu \;=\; \frac{\gamma_\mu\not{k}\gamma_\nu}{k^2+i0} \;+\; \frac{\gamma_\mu\not{k}(m-\not{p})\not{k}\gamma_\nu}{[k^2+i0]^2}\,. \quad \text{(S.15)}$$

Multiplying this expression by the photon propagator (2), we obtain

$$\begin{aligned}
\text{integrand} &= \frac{\gamma_\mu \not{k} \gamma^\mu}{[k^2+i0]^2} + \frac{\gamma_\mu \not{k}(m-\not{p}) \not{k}\gamma^\mu}{[k^2+i0]^3} + (\xi-1)\frac{\not{k}\not{k}\not{k}}{[k^2+i0]^3} + (\xi-1)\frac{\not{k}\not{k}(m-\not{p})\not{k}\not{k}}{[k^2+i0]^4} \\
&= \frac{-2\not{k}}{[k^2+i0]^2} + \frac{4mk^2 + 2\not{k}\not{p}\not{k}}{[k^2+i0]^3} + (\xi-1)\frac{\not{k}}{[k^2+i0]^2} + (\xi-1)\frac{m-\not{p}}{[k^2+i0]^2} \\
&= (\xi-3)\frac{\not{k}}{[k^2+i0]^2} + (\xi+3)\frac{m}{[k^2+i0]^2} + (1-\xi)\frac{\not{p}}{[k^2+i0]^2} + 2\frac{\not{k}\not{p}\not{k}}{[k^2+i0]^3}\,.
\end{aligned} \tag{S.16}$$

Moreover, in the context of a Lorentz-invariant momentum integral, the first term on the bottom line here integrates to zero, while in the numerator of the last term $k^\mu k^\nu \cong g^{\mu\nu}k^2/4$ and hence

$$2\not{k}\not{p}\not{k} = 4(kp)\not{k} - 2k^2\not{p} \cong 4\not{p}\times\frac{k^2}{4} - 2\not{p}k^2 = -k^2\times\not{p}. \tag{S.17}$$

Thus,

$$\text{integrand} \cong (\xi+3)\frac{m}{[k^2+i0]^2} + (1-\xi-1)\frac{\not{p}}{[k^2+i0]^2} \tag{S.18}$$

and therefore

$$\Sigma^{1\,\text{loop}}_{\text{div}} = e^2\big[(\xi+3)m - \xi\not{p}\big]\times\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\frac{-i}{[k^2+i0]^2}\,. \tag{S.19}$$

The integral here seems to have both UV and IR divergences in 4 dimensions, but the IR divergence is an artefact of the $1/\not{k}$ expansion (S.13) which does not work for small momenta. On the other hand, the UV divergence is genuine,

$$\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\frac{-i}{[k^2+i0]^2} = \frac{+1}{16\pi^2}\times\left(\frac{1}{\epsilon} + \text{const } \textbf{or } \log\Lambda^2 + \text{const}\right), \tag{S.20}$$

therefore

$$\Sigma^{1\,\text{loop}}(\not{p}) = \frac{e^2}{16\pi^2}\times\big[(3+\xi)m - \xi\not{p}\big]\times\left(\frac{1}{\epsilon}\ \textbf{or}\ \log\Lambda^2\right) + \text{finite}(\not{p}). \tag{S.21}$$

This divergence must be canceled by the QED counterterms δ_2 and δ_m according to

$$\Sigma^{\text{net}}(\not{p}) = \Sigma^{\text{loops}}(\not{p}) + \delta_m - \delta_2\times\not{p}, \tag{S.22}$$

hence at the one-loop level

$$\delta_m = -\frac{\alpha}{4\pi} \times (3+\xi)m \times \left(\frac{1}{\epsilon} \textbf{ or } \log\Lambda^2\right) + \text{finite}, \tag{S.23}$$

$$\delta_2 = -\frac{\alpha}{4\pi} \times \xi \times \left(\frac{1}{\epsilon} \textbf{ or } \log\Lambda^2\right) + \text{finite}. \tag{S.24}$$

Problem **1**(c):

In the MS renormalization scheme the counterterms (S.23) and (S.24) have no finite parts,

$$\delta_2 = \frac{1}{\epsilon} \times \frac{-\alpha\xi}{4\pi} + O(\alpha^2) \tag{S.25}$$

and

$$\delta_m = \frac{1}{\epsilon} \times \frac{-\alpha(3+\xi)m}{4\pi} + O(\alpha^2 m), \tag{S.26}$$

i.e.,

$$\hat{\delta} = \frac{1}{\epsilon} \times \frac{-\alpha(3+\xi)}{4\pi} + O(\alpha^2). \tag{S.27}$$

Plugging these counterterms into eq. (S.10) we immediately obtain

$$\gamma_m = \frac{\alpha}{4\pi} \times \left[-(3+\xi) + \xi\right] + O(\alpha^2) = -\frac{3\alpha}{4\pi} + O(\alpha^2). \tag{S.28}$$

Note that the gauge dependence of the δ_2 and δ_m counterterms cancels out and the anomalous dimension (S.28) of the electron's mass comes out to be gauge invariant.

Problem **1**(d):

Evolution of the renormalized electron's mass with energy is given by eq. (1). Integrating this equation, we obtain

$$\log \frac{m_e(M_w)}{m_e(m_e)} = \int\limits_{\log m_e}^{\log M_w} \gamma_m(\alpha(E))\, d\log E. \tag{S.29}$$

At the one-loop level, the anomalous dimension of the mass is given by eq. (S.28), hence

$$\log \frac{m_e(M_w)}{m_e(m_e)} \approx -\frac{3}{4\pi} \int\limits_{\log m_e}^{\log M_w} \alpha(E)\, d\log E. \tag{S.30}$$

The rest of this exercise is numerics. Between the electron mass scale $m_e = 511$ keV and the weak scale M_w — which we identify with the Z^0 mass $M_Z = 91$ GeV — the EM coupling changes from

$$\alpha(m_e) \approx \alpha(0) \approx \frac{1}{137.03} \tag{S.31}$$

to

$$\alpha(M_z) \approx \frac{1}{129.65} \tag{S.32}$$

This change is only 5%, so to the first approximation we may ignore it. In other words, we approximate $\alpha \approx \text{const} = 1/135$ (average value), which leads to

$$\log \frac{m_e(M_w)}{m_e(m_e)} \approx -\frac{3\alpha}{4\pi} \times \log \frac{M_Z}{m_e} \approx -0.067.$$

Consequently,

$$m_e(M_z) \approx m_e^{\text{phys}} \times (1 - 0.067) = 477 \text{ keV}. \tag{S.33}$$

Problem **3**(a):

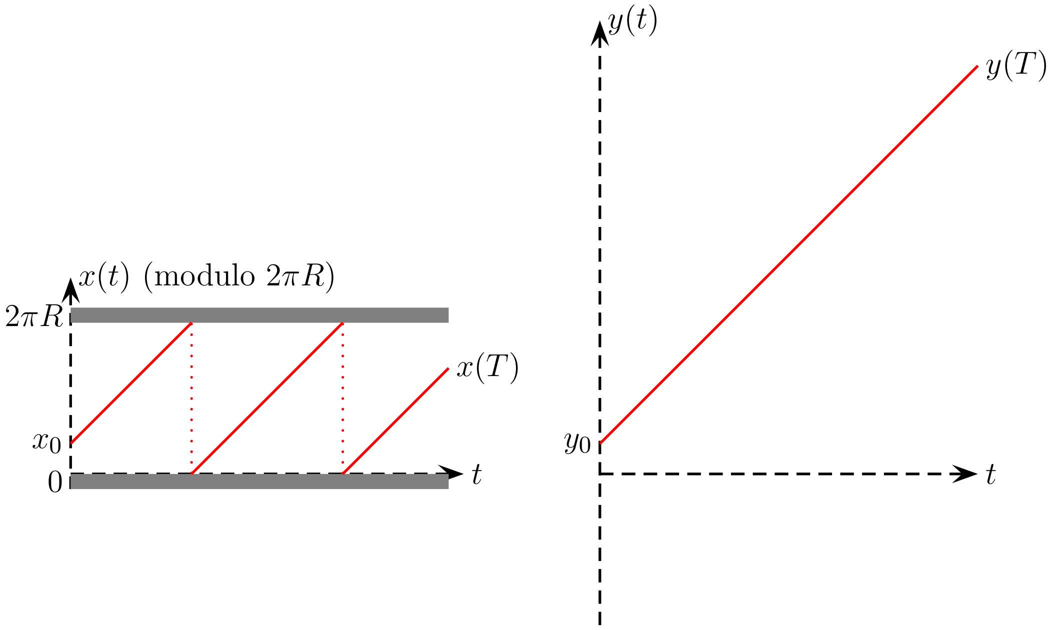
The difference between a circle and a straight line is that on a circle the path of a particle going from point x_0 to point x' does not need to be 'straight' but may wrap around the whole circle one or more times. Indeed, let us compare a particle moving on a circle according to $x(t)$ (modulo $2\pi R$) with a particle moving on an infinite line according to $y(t)$. If the two particles have exactly the same velocities at all times,

$$\frac{dx}{dt} \equiv \frac{dy}{dt} \tag{S.34}$$

and similar initial positions $x_0 = y_0$ (according to some coordinate systems) at time $t = 0$, then after time T one generally has

$$y(T) = x(T) + 2\pi R \times n \tag{S.35}$$

for some integer $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ because the $x(y)$ path may wrap around the circle n times while the $y(t)$ path may not wrap. For example, the two paths depicted below have same (constant) velocities and begin at $y_0 = x_0$ but end at $y(T) = x(T) + 2\pi R \times 2$:



It is easy to see that the paths $x(t)$ (modulo $2\pi R$) and $y(t)$ (modulo nothing) are in one-to-one correspondence with each other, provided we restrict the initial point y_0 of the particle on the infinite line to a particular interval of length $L = 2\pi R$, say $0 \le y_0 < 2\pi R$. Consequently, in the path integral for the particle on the circle

$$\iint_{x(t=0)=x_0\,(\mathrm{mod}\,L)}^{x(t=T)=x'\,(\mathrm{mod}\,L)} \mathcal{D}'[x(t)\,(\mathrm{mod}\,L)] \;=\; \sum_{n=-\infty}^{+\infty} \iint_{y(t=0)=x_0}^{y(t=T)=x'+nL} \mathcal{D}'[y(t)]\,. \tag{S.36}$$

Furthermore, in the absence of potential energy, the circle path $x(t)\,(\mathrm{mod}\,L)$ and the corresponding ∞ line path $y(t)$ have equal actions

$$S[x(t)\,(\mathrm{mod}\,L)] \;=\; S[y(t)] \;=\; \int_0^T dt\,\left[\tfrac{M}{2}\dot{x}^2 \;=\; \tfrac{M}{2}\dot{y}^2\right], \tag{S.37}$$

and therefore

$$\begin{aligned} U_{\mathrm{circle}}(x';x_0) \;&=\; \iint_{x(t=0)=x_0\,(\mathrm{mod}\,L)}^{x(t=T)=x'\,(\mathrm{mod}\,L)} \mathcal{D}'[x(t)\,(\mathrm{mod}\,L)]\, e^{iS[x(t)\,(\mathrm{mod}\,L)]/\hbar} \\ &=\; \sum_{n=-\infty}^{+\infty} \iint_{y(t=0)=x_0}^{y(t=T)=x'+nL} \mathcal{D}'[y(t)]\, e^{iS[y(t)]/\hbar} \\ &=\; \sum_{n=-\infty}^{+\infty} U_{\infty\,\mathrm{line}}(y' = x' + nL; y_0 = x_0). \end{aligned} \tag{1}$$

$\mathcal{Q.E.D.}$

Problem **3**(b):

For a free particle living on an infinite line the evolution kernel is given by

$$U_{\infty\,\mathrm{line}}(y';y_0) \;=\; \sqrt{\frac{M}{2\pi i\hbar T}} \times \exp\left(\frac{i}{\hbar}S_{\mathrm{classical}} \;=\; \frac{i}{\hbar}\,\frac{M(x'-x_0)^2}{2T}\right), \tag{3}$$

hence according to eq. (1), a particle on a circle has kernel

$$U_{\mathrm{circle}}(x';x_0) \;=\; \sqrt{\frac{M}{2\pi i\hbar T}} \times \sum_{n=-\infty}^{+\infty} \exp\left(\frac{iM}{2\hbar T}\,(x'-x_0+nL)^2\right). \tag{S.38}$$

To evaluate this sum, we use Poisson re-summation formula (2), which gives

$$\sum_{n=-\infty}^{+\infty} \exp\left(\frac{iM}{2\hbar T}(x'-x_0+nL)^2\right) = \sum_{\ell=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu \, \exp\left(\frac{iM}{2\hbar T}(x'-x_0+\nu L)^2\right) \times e^{2\pi i\ell\nu}. \quad \text{(S.39)}$$

Rearranging the exponential, we have

$$\frac{iM}{2\hbar T}(x'-x_0+\nu L)^2 + 2\pi i\ell\nu = \frac{iML^2}{2\hbar T}\left(\nu + \frac{x'-x_0}{L} + \frac{2\pi\ell\hbar T}{ML^2}\right) - 2\pi i\ell\,\frac{x'-x_0}{L} - \frac{i\hbar T(2\pi\ell)^2}{ML^2}, \quad \text{(S.40)}$$

and therefore

$$\int_{-\infty}^{+\infty} d\nu \, \exp\left(\frac{iM}{2\hbar T}(x'-x_0+\nu L)^2\right) \times e^{2\pi i\ell\nu} = \sqrt{\frac{2\pi i\hbar T}{ML^2}} \times \exp\left(-2\pi i\ell\,\frac{x'-x_0}{L} - \frac{(2\pi\ell)^2 i\hbar T}{ML^2}\right). \quad \text{(S.41)}$$

Consequently,

$$\begin{aligned} U_{\text{circle}}(x';x_0) &= \sqrt{\frac{M}{2\pi i\hbar T}} \times \sqrt{\frac{2\pi i\hbar T}{ML^2}} \times \sum_{\ell=-\infty}^{+\infty} \exp\left(-2\pi i\ell\,\frac{x'-x_0}{L} - \frac{(2\pi\ell)^2 i\hbar T}{ML^2}\right) \\ &= \frac{1}{L}\sum_{\ell=-\infty}^{+\infty} e^{ip(x'-x_0)/\hbar} \times e^{-iTE/\hbar} \end{aligned} \quad \text{(S.42)}$$

where

$$p = -\frac{2\pi\hbar\ell}{L} = -\frac{\hbar\ell}{R} \quad \text{and} \quad E = \frac{p^2}{2M}. \quad \text{(S.43)}$$

Problem **3**(c): This is obvious from eqs. (S.42) and (S.43).