Problem **1**(a):

Let us evaluate the trace of the Casimir operator C_2 over an irreducible multiplet (r). On one hand,

$$\begin{aligned} \mathrm{tr}_{(r)}\left(C_2 \stackrel{\mathrm{def}}{=} \sum_a T^a T^a\right) &= \sum_a \mathrm{tr}_{(r)}\left(T^a T^a\right) = \sum_a \mathrm{tr}\left(T^a_{(r)} T^b_{(r)}\right) \\ \langle\!\langle \text{by eq. (1)} \rangle\!\rangle \quad &= \sum_a R(r) \times (\delta^{aa} = 1) = R(r) \times \dim(G) \end{aligned} \tag{S.1}$$

where $\dim(G) \stackrel{\mathrm{def}}{=} \dim(\mathrm{Adj}(G))$ is the number of the generators of the symmetry group G — which is also the dimension of the adjoint representation of G, hence the notation. On the other hand,

$$\mathrm{tr}_{(r)}(C_2) = \mathrm{tr}_{(r)}\left(C_2\big|_{(r)}\right) = \mathrm{tr}\left(C(r) \times \mathbf{1}_{(r)}\right) = C(r) \times \dim(r). \tag{S.2}$$

Together, eqs. (S.1) and (S.2) immediately imply eq. (3), *quod erat demonstrandum.*

For the special case of $G = SU(2)$, an irreducible multiplets of isospin I has $C = \mathbf{I}^2 = I(I+1)$ and dimension $2I + 1$, hence

$$R(I) = C(I) \times \frac{\dim(I)}{\dim(G)} = I(I+1) \times \frac{2I+1}{3}. \tag{S.3}$$

Problem **1**(b):

Unlike the Casimir value $C(r)$, the index $R(r)$ is well defined for any complete multiplet (r), irreducible or otherwise. For a reducible multiplet

$$(r) = \bigoplus_{i=1}^{n} (r_i) \equiv (r_1) \oplus (r_2) \oplus \cdots \oplus (r_n)$$

one has

$$\begin{aligned} \mathrm{tr}_{(r)}\left(T^a T^b\right) &= \mathrm{tr}\left(T^a T^b\Big|_{\bigoplus_{i=1}^n (r_i)}\right) = \sum_{i=1}^{n} \mathrm{tr}\left(T^a T^b\Big|_{(r_i)}\right) \\ &= \sum_{i=1}^{n} \left(R(r_i) \times \delta^{ab}\right) = \delta^{ab} \times \sum_{i=1}^{n} R(r_i) \end{aligned} \tag{S.4}$$

and thus

$$R(r) \;=\sum_{i=1}^{n} R(r_i). \tag{S.5}$$

In particular, a reducible multiplet

$$(r) \;=\; \bigoplus_{i=1}^{n} (I_i)$$

of the isospin group $SU(2)$ has index

$$R(r) \;=\sum_{i=1}^{n} \tfrac{1}{3} I(I+1)(2I+1). \tag{S.6}$$

Now consider a bigger symmetry group G which contains the 'isospin' $SU(2)$ as a subgroup. Then any complete multiplet (r) of G is automatically a complete multiplet of the $SU(2) \subset G$. However, irreducible multiplets of G usually become reducible from the $SU(2)$ point of view, $(r) = (I_1) \oplus (I_2) \oplus \cdots \oplus (I_n)$; for example, the adjoint multiplet of $SU(3)$ decomposes into $(0) \oplus (\frac{1}{2}) \oplus (\frac{1}{2}) \oplus (1)$ of the $SU(2) \subset SU(3)$. Let T^1, T^2, and T^3 be generators of the $SU(2)$ subgroup of G. Then according to eq. (S.6),

$$\text{for } a, b = 1, 2, 3, \quad \text{tr}_{(r)} \left(T^a T^b \right) \;=\; \delta^{ab} \times \sum_{i=1}^{n} \tfrac{1}{3} I(I+1)(2I+1). \tag{S.7}$$

Now, let us suppose that the Lie group G is *simple*, that is, all its generators are related to each other by the symmetry G itself. In this case, for any complete multiplet (r) of G

$$\text{tr}_{(r)} \left(T^a T^b \right) \;=\; R(r) \times \delta^{ab}, \quad \text{same } R(r)\ \forall a, b = 1, \ldots, \dim(G). \tag{S.8}$$

Combining this formula with eq. (S.7) we immediately obtain

$$R(r) \;=\sum_{i=1}^{n} \tfrac{1}{3} I(I+1)(2I+1), \tag{4}$$

quod erat demonstrandum.

Caveat: We have silently assumed that $T^{1,2,3}$ have the same normalization as generators of G as the have as generators of the $SU(2) \subset G$. This assumption is correct for the $SU(2) \subset SU(N)$ discussed in parts (c) and (d) of this problem, but it would fail for a different (*i.e.*, inequivalent) $SU(2)$ subgroup. In general, properly normalized $SU(2) \subset G$ generators $I^{1,2,3}$ are related to the properly normalized generators of G as

$$I^a = T^{(a)} \times \sqrt{k} \tag{S.9}$$

where $T^{(1)}$, $T^{(2)}$, and $T^{(3)}$ are 3 generators of G which happen to satisfy $[T^{(a)}, T^{(b)}] = i\epsilon^{abc}T^{(c)}/\sqrt{k}$. The k here is always a positive integer; it's called *the level of embedding of the $SU(2)$ into G*. For example, consider the $SU(2)$ subgroup of $SU(3)$ which acts on the fundamental triplet as a real $SO(3)$ rotation. This subgroup is generated by the

$$I^1 = \sqrt{4}\times T^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +i \\ 0 & -i & 0 \end{pmatrix}, \quad I^2 = \sqrt{4}\times T^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{pmatrix}, \quad I^3 = \sqrt{4}\times T^2 = \begin{pmatrix} 0 & +i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{S.10}$$

(note $T^a = \frac{1}{2}\lambda^a$), so its embedding level is $k = 4$.

When you decompose a multiplet (r) of G into irreducible multiplets of an $SU(2)$ subgroup, you should take into account the level at which this $SU(2)$ is embedded into G. As written, eq. (4) works only for the $k = 1$ subgroups; for other embedding levels,

$$R(r) = \frac{1}{k}\sum_{i=1}^{n} \tfrac{1}{3}I(I+1)(2I+1). \tag{S.11}$$

Note that the decomposition of the G multiplet (r) into $SU(2)$ multiplets depends on the $SU(2)$ embedding into G. For example, under the $k = 1$ subgroup $SU(2) \subset SU(3)$

$$\text{triplet} = (\tfrac{1}{2}) \oplus (0), \qquad \text{octet} = (1) \oplus (\tfrac{1}{2}) \oplus (\tfrac{1}{2}) \oplus (0), \tag{S.12}$$

while under the $k = 4$ subgroup (S.10)

$$\text{triplet} = (1), \qquad \text{octet} = (1) \oplus (2). \tag{S.13}$$

In both cases, eq. (S.11) produces the same index R for each $SU(3)$ multiplet, *e.g.* $R(\text{triplet}) = \frac{1}{2}$, $R(\text{octet}) = 3$, but only if you remember the $1/k$ factor in front of the sum.

Problem $\mathbf{1}$(c):

From the $SU(2) \subset SU(N)$ point of view, the fundamental representation $\mathbf{N}$ of the $SU(N)$ decomposes into one doublet plus $(N-2)$ singlets,

$$\mathbf{N} \;=\; \mathbf{2} \;+\; (N-2)\times\mathbf{1} \;\equiv\; (I=\tfrac{1}{2}) \;+\; (N-2)\times(I=0), \tag{S.14}$$

hence according to eq. (4),

$$R(\mathbf{N}) \;=\; R(I=\tfrac{1}{2}) \;+\; (N-2)\times R(I=0) \;=\; \tfrac{1}{2} \;+\; (N-2)\times 0 \;=\; \tfrac{1}{2}$$

and consequently

$$C(\mathbf{N}) \;=\; R(\mathbf{N})\times\frac{\dim(G)}{\dim(\mathbf{N})} \;=\; \frac{1}{2}\times\frac{N^2-1}{N} \;=\; \frac{N^2-1}{2N} \tag{4}$$

Now consider the adjoint representation of the $SU(N)$. Let us form a tensor product of the fundamental representation $\mathbf{N}$ and the conjugate (anti-fundamental) representation $\overline{\mathbf{N}}$. Given the transformation laws

$$\begin{aligned} \Psi \;&\to\; U\Psi, \qquad &i.e.\quad \Psi'_j \;&=\; U_j^{\,k}\Psi_k\,, \\ \overline{\Psi} \;&\to\; \overline{\Psi}U^\dagger, \qquad &i.e.\quad \overline{\Psi}'^{\ell} \;&=\; \overline{\Psi}^m\, U^{*\ell}_{\;m}\,, \end{aligned}$$

it follows that the tensor product is a hermitian $N\times N$ matrix Φ_j^k which transforms as

$$\Phi' \;=\; U\Phi U^\dagger \qquad i.e.\quad \Phi_j'^{\ell} \;=\; U_j^{\,k}\Phi_k^m U^{*\ell}_{\;m}\,. \tag{5}$$

This matrix is a reducible multiplet $\text{Adj}+\mathbf{1}$ of the $SU(N)$: The trace $\text{tr}(\Phi)$ is an invariant singlet, while the traceless part $\Phi_i^{\,j}-\delta_i^j\times\text{tr}(\Phi)/N$ forms the adjoint multiplet (*cf.* homework set#2 back in September). In other words,

$$\mathbf{N}\otimes\overline{\mathbf{N}} \;=\; \text{Adj}\oplus\mathbf{1} \tag{S.15}.$$

In $SU(2)$ $\overline{\mathbf{2}}=\mathbf{2}$, so from the $SU(2)\subset SU(N)$ point of view, both the fundamental and the anti-fundamental multiplets of the $SU(N)$ decompose into similar sets of one doublet and $N-2$

singlets. Therefore,

$$\begin{aligned}
[\text{Adj} \ + \ \mathbf{1}]_{SU(N)} \ &= \ \big[\mathbf{N}\otimes\overline{\mathbf{N}}\big]_{SU(N)} \\
&= \ [\mathbf{2} \ + \ (N-2)\times\mathbf{1}]_{SU(2)} \ \otimes \ [\mathbf{2} \ + \ (N-2)\times\mathbf{1}]_{SU(2)} \\
&= \ \big[(\mathbf{2}\otimes\mathbf{2}) \ + \ 2(N-2)\times(\mathbf{2}\otimes\mathbf{1}) \ + \ (N-2)^2\times(\mathbf{1}\otimes\mathbf{1})\big]_{SU(2)} \\
&= \ \big[\mathbf{3} \ + \ \mathbf{1} \ + \ 2(N-2)\times\mathbf{2} \ + \ (N-2)^2\times\mathbf{1}\big]_{SU(2)}\,, \\
\textit{i.e.}\quad [\text{Adj}]_{SU(N)} \ &= \ \big[\mathbf{3} \ + \ 2(N-2)\times\mathbf{2} \ + \ (N-2)^2\times\mathbf{1}\big]_{SU(2)}\,,
\end{aligned} \tag{S.16}$$

and consequently

$$\begin{aligned}
R(\text{Adj}) \ &= \ I(\mathbf{3}) \ + \ 2(N-2)\times I(\mathbf{2}) \ + \ (N-2)^2\times I(\mathbf{1}) \\
&= \ 2 \ + \ 2(N-2)\times\tfrac{1}{2} \ + \ (N-2)^2\times 0 \ = \ N.
\end{aligned} \tag{S.17}$$

Finally,

$$C(G) \ \stackrel{\text{def}}{=} \ C(\text{Adj}(G)) \ = \ R(\text{Adj})\times\frac{\dim(G)}{\dim(G)} \ = \ R(\text{Adj}) \ = N. \tag{S.18}$$

Problem **1**(d):

Consider the two-index symmetric tensor $S_{(ij)}$ representation of the $SU(N)$ symmetry group. Denote the index $i=\alpha$ if $i=1,2$ or $i=\mu$ if $i=3,4,\ldots,N$ and likewise $j=\beta$ if $j=1,2$ and $j=\nu$ if $j=3,4,\ldots,N$. Thus, the complete set of independent $S_{(ij)}$ decomposes into $S_{(\alpha\beta)}$, $S_{\alpha,\mu}\equiv S_{\mu,\alpha}$ and $S_{(\mu\nu)}$. The $SU(2)\subset SU(N)$ acts on indices $\alpha,\beta=1,2$ and ignores indices $\mu,\nu=3,4,\ldots,N$, so from the $SU(2)$ point of view, $S_{(\alpha\beta)}$ is a triplet, $S_{\alpha,\mu}$ are $N-2$ separate doublets, and $S_{(\mu\nu)}$ are $(N-2)(N-1)/2$ singlets. Consequently,

$$\begin{aligned}
R(S) \ &= \ R(\mathbf{3}) \ + \ (N-2)\times R(\mathbf{2}) \ + \ \tfrac{1}{2}(N-1)(N-2)\times R(\mathbf{1}) \\
&= \ 2 \ + \ (N-2)\times\tfrac{1}{2} \ + \ \tfrac{1}{2}(N-1)(N-2)\times 0 \ = \ \tfrac{1}{2}(N+2),
\end{aligned} \tag{S.19}$$

and hence

$$C(S) = \ R(S)\times\frac{\dim(G)}{\dim(S)} \ = \ \frac{N+2}{2}\times\frac{N^2-1}{\frac{1}{2}N(N+1)} \ = \ \frac{N^2+N-2}{N}\,. \tag{S.20}$$

Similarly, the two-index anti-symmetric tensor $A_{[ij]}$ decomposes into $A_{[\alpha\beta]}$, $A_{\alpha,\mu}$, and $A_{[\mu\nu]}$. In $SU(2)$, the $A_{[\alpha\beta]}$ is equivalent to the trivial singlet $A\times\epsilon_{[\alpha\beta]}$, the $A_{\alpha,\mu}$ are $N-2$ doublets, and

$A_{[\mu\nu]}$ are $(N-2)(N-3)/2$ singlets. Altogether

$$(A) \;=\; (N-2)\times\mathbf{2} \;+\; \text{singlets},$$

therefore

$$R(A) \;=\; (N-2)\times\tfrac{1}{2} \;+\; 0 \;=\; \tfrac{1}{2}(N-2) \tag{S.21}$$

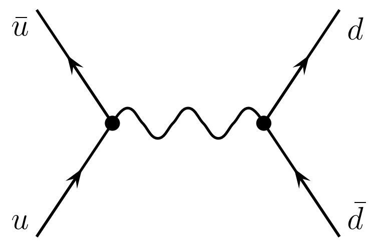
and

$$C(A) = \; R(A)\times\frac{\dim(G)}{\dim(A)} \;=\; \frac{N-2}{2}\times\frac{N^2-1}{\frac{1}{2}N(N-1)} \;=\; \frac{N^2-N-2}{N}\,. \tag{S.22}$$

Problem **2**:

At the tree level of QCD,

$$i\mathcal{M}(u\bar{u}\to d\bar{d}) \;=$$



(S.23)

$$=\; \frac{ig^2}{s}\times\bar{v}(\bar{u})\gamma^\mu u(u)\,(T^a)^i{}_j\times\bar{u}(d)\gamma_\mu v(d)\,(T^a)^k{}_\ell$$

where $s=E^2_{\text{c.m.}}$, the quarks and the antiquarks have color indices i,j,k,ℓ, and the virtual gluon has gauge index a in the adjoint representation; the summation over a is implicit. Except for the gauge indices, the $u\bar{u}\to d\bar{d}$ process in QCD is completely analogous to the $e^-e^+\to\mu^-\mu^+$ pair production in QED. In particular, summing / averaging $|\mathcal{M}|^2$ over the fermion's spins yields

$$\begin{aligned}\tfrac{1}{4}\sum_{\text{all spins}}\left|\bar{v}(\bar{u})\gamma^\mu u(u)\,\bar{u}(d)\gamma_\mu v(d)\right|^2 \;&\approx\; \tfrac{1}{4}\,\text{tr}(\not{p}_{\bar{u}}\gamma^\mu\,\not{p}_u\gamma^\nu)\times\text{tr}(\not{p}_{\bar{d}}\gamma_\mu\,\not{p}_d\gamma_\nu)\\ &=\; 2(t^2+u^2) \;=\; s^2(1+\cos^2\theta_{\text{c.m.}})\end{aligned} \tag{S.24}$$

where the approximation is neglecting the quark masses m_u and m_d.

The new part of this exercise is summing / averaging over the color indices. By hermiticity of the Lie Algebra matrices T^a, we have

$$\Big((T^a)^i_{\ j}\,(T^a)^k_{\ \ell}\Big)^* \;=\; (T^a)^j_{\ i}\,(T^a)^\ell_{\ k} \;=\; (T^b)^j_{\ i}\,(T^b)^\ell_{\ k} \tag{S.25}$$

— note the implicit summation over a or b — and hence

$$\begin{aligned}\sum_{i,j,k,\ell}\Big|(T^a)^i_{\ j}\,(T^a)^k_{\ \ell}\Big|^2 &= \sum_{i,j,k,\ell}(T^a)^i_{\ j}\,(T^a)^k_{\ \ell}\times(T^b)^j_{\ i}\,(T^b)^\ell_{\ k}\\ &= \sum_{ij}(T^a)^i_{\ j}\,(T^b)^j_{\ i}\times\sum_{k,\ell}(T^a)^k_{\ \ell}\,(T^b)^\ell_{\ k}\\ &= \mathrm{tr}(T^aT^b)\times\mathrm{tr}(T^aT^b)\end{aligned} \tag{S.26}$$

For the moment, let us consider 'quarks' belonging to some generic multiplet (r) of some generic gauge group G. In such a generic case, $\mathrm{tr}(T^aT^b)=R(r)\times\delta^{ab}$ where $R(r)$ is the index of the quark multiplet, *cf.* problem 1, and hence

$$\sum_{a,b}\mathrm{tr}(T^aT^b)\times\mathrm{tr}(T^aT^b) \;=\; R^2(r)\times\sum_{a,b}\delta^{ab}\delta^{ab} \;=\; R^2(r)\times\dim(G). \tag{S.27}$$

Substituting this formula into eq. (S.26) then gives

$$\sum_{i,j,k,\ell}\Big|\sum_a(T^a)^i_{\ j}\,(T^a)^k_{\ \ell}\Big|^2 \;=\; R^2(r)\times\dim(G),$$

or, for the average over the initial 'colors' i and j,

$$\frac{1}{\dim^2(r)}\sum_{i,j}\sum_{k,\ell}\Big|\sum_a(T^a)^i_{\ j}\,(T^a)^k_{\ \ell}\Big|^2 \;=\; \frac{R^2(r)\,\dim(G)}{\dim^2(r)} \;=\; \frac{C^2(r)}{\dim(G)}\,. \tag{S.28}$$

Specializing to the 'quarks' in the fundamental representation of an $SU(N)$ gauge group, we have $R(r)=\frac{1}{2}$, $\dim(r)=N$ and $\dim(G)=N^2-1$, hence eq. (S.28) evaluates to $(N^2-1)/(4N^2)$; for the actual QCD $N=3$ and the color sum / average (S.28) gives $2/9$.

Altogether, $|\mathcal{M}|^2$ summed / averaged over both spins and colors of all the fermions is

$$\frac{2}{9}\times g^4(1+\cos^2\theta_{\text{c.m.}}) \tag{S.29}$$

and hence the total cross section

$$\sigma(u\bar{u}\to d\bar{d}) \;=\; \frac{8\pi\alpha_{\text{QCD}}^2}{27E_{\text{c.m.}}^2}\,. \tag{S.30}$$

<u>Problem **3**(a)</u>:

Classically,

$$\mathcal{L} \;=\; \mathcal{L}_{\text{YM}} \;+\; D^\mu\Phi^\dagger D_\mu\Phi \;-\; V(\Phi^\dagger,\Phi) \tag{S.31}$$

where

$$D_\mu\Phi_i \;=\; \partial_\mu\Phi_i \;+\; igA_\mu^a(T_{(r)}^a)_i^{\,j}\Phi_j\,, \qquad D^\mu\Phi^{*i} \;=\; \partial^\mu\Phi^{*i} \;-\; igA^{a\mu}\Phi^{*j}(T_{(r)}^a)_j^{\,i}, \tag{S.32}$$

and $V(\Phi^\dagger,\Phi)$ is some kind of a G–invariant potential. For scalars in the fundamental multiplet of an $SU(N)$ gauge group, there is only one independent G–invariant function of Φ_i, namely $\Phi^\dagger\Phi\equiv\Phi^{*i}\Phi_i$, hence $V=V(\Phi^\dagger\Phi)$. Furthermore, for renormalizability's sake, V should be a polynomial of degree 4 (or less), thus

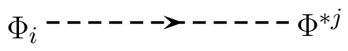
$$V(\Phi) \;=\; m^2(\Phi^\dagger\Phi) \;+\; \tfrac{1}{8}\lambda(\Phi^\dagger\Phi)^2. \tag{S.33}$$

For more complicated representations of the gauge group the scalar potential may include additional, independent G–invariant terms. For example, for the anti-symmetric tensor multiplet $\Phi_{[ij]}$ we may have

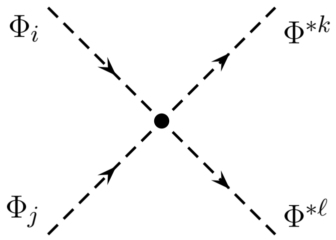
$$V \;=\; \tfrac{1}{2}m^2\Phi^{*ij}\Phi_{ij} \;+\; \tfrac{1}{32}\lambda(\Phi^{*ij}\Phi_{ij})^2 + \tfrac{1}{8}\lambda'\,\Phi^{*ij}\Phi_{jk}\Phi^{*k\ell}\Phi_{\ell i}\,. \tag{S.34}$$

But the details of the scalar potential are not germane to this problem, so I'll assume it has form (S.33) regardless of (r).

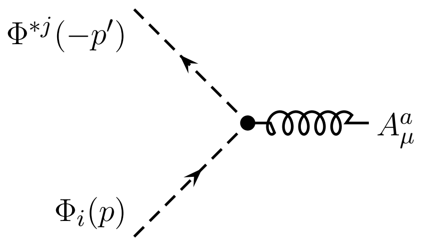
In the quantum field theory, the net Lagrangian comprises the classical terms (S.31) plus the ghost Lagrangian, the gauge fixing terms, and the whole slew of counterterms. Generally, all terms pertaining only to the gauge fields and ghost fields have exactly the same form as in the fermionic QCD, so the 'gluon' propagator, the three-gluon and the four-gluon vertices, the ghost propagator and the ghost-gluon vertex are no different from the notes I distributed in class and I don't need to repeat them here. The Feynman rules peculiar to the scalar QCD are those pertaining to the scalar fields, thus



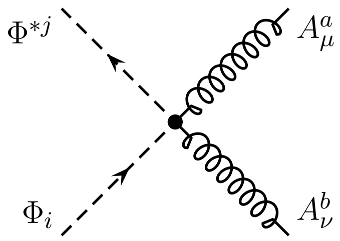
$$= \frac{i\delta_i^j}{p^2 - m^2 + i0},$$



$$= -i\lambda(\delta_i^k \delta_j^\ell + \delta_i^\ell \delta_j^k),$$



$$= ig(p+p')_\mu (T_{(r)}^a)^j{}_i,$$



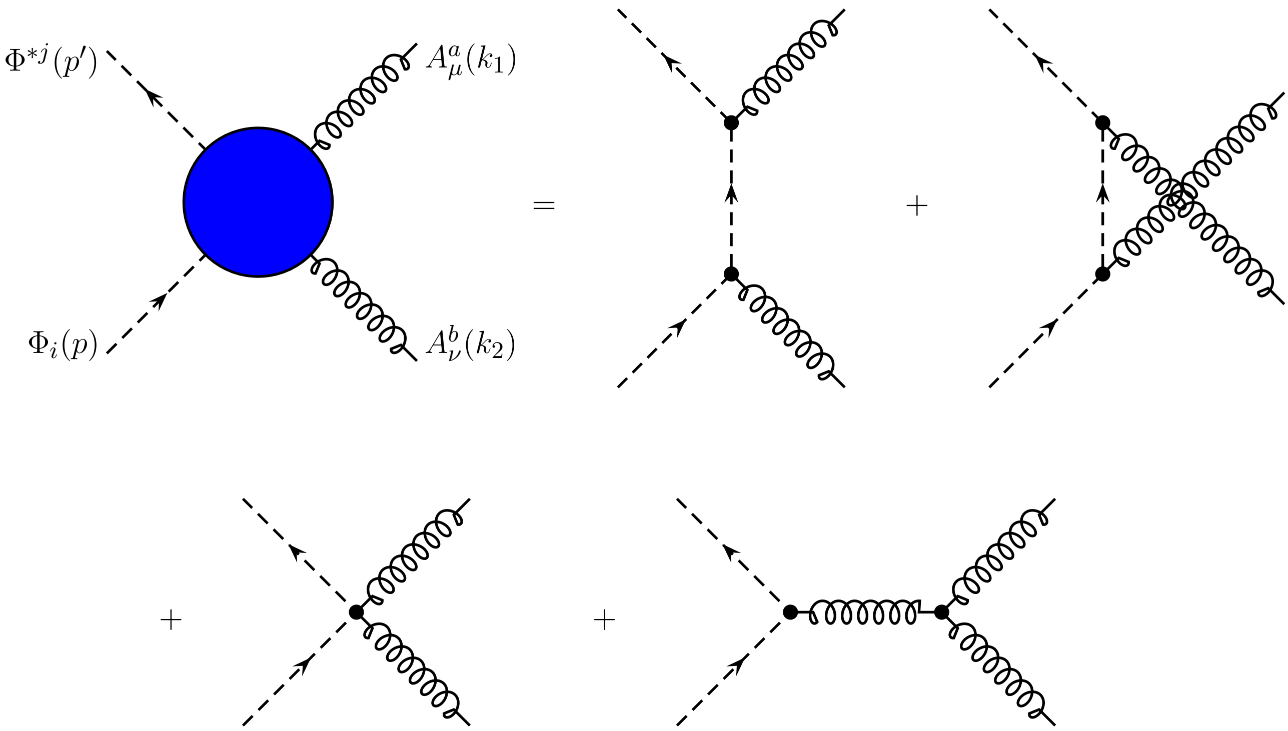
$$= -ig^2 g_{\mu\nu} \{T_{(r)}^a, T_{(r)}^b\}^j{}_i.$$

Note the anticommutator of the group generators in the two-scalar two-gluon vertex: It follows from permutations of the two gluon lines.

In addition, there are also counterterm vertices involving the scalar fields, but they not germane to the present exercise.

Problem **3**(b):

At the tree level there are four diagrams for the $\Phi\Phi^* \to gg$ annihilation process,



(S.35)

so the net tree-level amplitude is $\mathcal{M} = \mathcal{M}^{\mu\nu} \times e^*_{1\mu} e^*_{2\nu}$ where

$$\begin{aligned}
\mathcal{M}^{\mu\nu} \;=\; & \frac{g^2}{(p-k_1)^2 - m^2}\,(k_2 - 2p')^\nu (2p - k_1)^\mu \,(T^b T^a)^j{}_i \\
& +\; \frac{g^2}{(p-k_2)^2 - m^2}\,(k_1 - 2p')^\mu (2p - k_2)^\nu \,(T^a T^b)^j{}_i \\
& -\; g^2 g^{\mu\nu}\,\{T^a, T^b\}^j{}_i \\
& -\; \frac{ig^2}{(k_1+k_2)^2}\,(p - p')_\lambda \,(T^c)^j{}_i \\
& \qquad \times f^{abc}\big(g^{\mu\nu}(k_1 - k_2)^\lambda \;+\; g^{\nu\lambda}(2k_2 + k_1)^\mu \;+\; g^{\lambda\mu}(-2k_1 - k_2)^\nu\big).
\end{aligned} \tag{S.36}$$

Problem **3**(c):

Our task is to verify that the amplitude (S.36) satisfies

$$k_1^\mu e_2^\nu \mathcal{M}_{\mu\nu} \;=\; 0 \tag{S.37}$$

provided $e_2^\nu k_{2\nu} = 0$ and all external momenta are on shell. Let $\mathcal{M}^{\mu\nu}_{1,2,3,4}$ denote respectively the four terms on the right hand side of eq. (S.36). Then it is easy to show that for $p^2 = p'^2 = m^2$

$$\begin{aligned} k_{1\mu}\mathcal{M}_1^{\mu\nu} &= g^2(2p'-k_2)^\nu\,(T^bT^a)^j_{\;i}\,, \\ k_{1\mu}\mathcal{M}_2^{\mu\nu} &= g^2(2p-k_2)^\nu\,(T^aT^b)^j_{\;i}\,, \\ k_{1\mu}\mathcal{M}_3^{\mu\nu} &= -g^2k_1^\nu\,\{T^a,T^b\}^j_{\;i}\,, \end{aligned} \tag{S.38}$$

and therefore

$$\begin{aligned} k_{1\mu}(\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3)^{\mu\nu} &= g^2(2p'-k_2-k_1)^\nu\,(T^bT^a)^j_{\;i} \;+\; g^2(2p-k_1-k_2)^\nu\,(T^aT^b)^j_{\;i} \\ &= g^2(p-p')^\nu \times (-T^bT^a+T^aT^b)^j_{\;i} \\ &= g^2(p-p')^\nu \times if^{abc}(T^c)^j_{\;i}\,. \end{aligned} \tag{S.39}$$

At the same time, for $k_1^2 = k_2^2 = 0$ we have

$$\begin{aligned} &k_{1\mu}\times\Big(g^{\mu\nu}(k_1-k_2)^\lambda \;+\; g^{\nu\lambda}(2k_2+k_1)^\mu \;+\; g^{\lambda\mu}(-2k_1-k_2)^\nu\Big) \\ &\quad= (k_1+k_2)^2g^{\nu\lambda} \;+\; k_2^\nu k_2^\lambda \;-\; (k_1+k_2)^\nu(k_1+k_2)^\lambda \end{aligned} \tag{S.40}$$

and hence

$$k_{1\mu}\mathcal{M}_4^{\mu\nu} \;=\; ig^2(p-p')_\lambda\left[g^{\lambda\nu} \;+\; \frac{k_2^\lambda k_2^\nu}{(k_1+k_2)^2} \;+\; 0\right] \times if^{abc}(T^c)^j_{\;i}. \tag{S.41}$$

because on shell $(k_1+k_2)^\lambda \times (p-p')_\lambda = (p+p')^\lambda(p-p')_\lambda = p^2 - p'^2 = 0$.

Consequently, in light of eq. (S.39), most terms in $k_{1\mu}\mathcal{M}^{\mu\nu}$ cancel out, except for the second term in eq. (S.41), thus

$$k_{1\mu}\mathcal{M}^{\mu\nu} \;=\; g^2\,\frac{k_2(p-p')}{(k_1+k_2)^2} \times if^{abc}(T^c)^j_{\;i} \times k_2^\nu\,. \tag{S.42}$$

Similarly to the fermionic QCD discussed in class, this amplitude does not quite vanish but is proportional to the k_2^ν. Consequently, $k_{1\mu}\mathcal{M}^{\mu\nu}e^*_{2\nu} \;=\; 0$ when the second gluon is transversely polarized, $k_2 e_2 = 0$, but not if the other gluon's polarization is longitudinal. And this is in accordance to the weak form of Ward Identity: *On-shell amplitudes involving one longitudinal gluon vanish, but only if all the other gluons are transverse.*

Problem **4**(a):

Consider the amplitude (S.36): For transverse gauge bosons $(e_1^* k_1) = (e_2^* k_2) = 0$, we have

$$\begin{aligned}\mathcal{M} \equiv \mathcal{M}^{\mu\nu} e_{1\mu}^* e_{2\nu}^* = & -\frac{4g^2}{t-m^2}(e_1^* p)(e_2^* p') \times (T^b T^a)^j{}_i \\ & - \frac{4g^2}{u-m^2}(e_1^* p')(e_2^* p) \times (T^a T^b)^j{}_i \\ & - g^2 (e_1^* e_2^*) \times \{T^a, T^b\}^j{}_i \\ & - \frac{ig^2}{s}\Big[(u-t)(e_1^* e_2^*) + 2(e_1^* k_2)\big(e_2^*(p-p')\big) - 2(e_2^* k_1)\big(e_1^*(p-p')\big)\Big] \\ & \quad \times f^{abc}(T^c)^j{}_i\end{aligned} \tag{S.43}$$

where and s, t and u are Mandelstam's kinematic variables and

$$(u-t) = (p-p')_\lambda (k_1 - k_2)^\lambda.$$

Clearly,

$$\begin{aligned}(T^a T^b) &= \tfrac{1}{2}\{T^a, T^b\} + \tfrac{1}{2}[T^a, T^b], \\ (T^b T^a) &= \tfrac{1}{2}\{T^a, T^b\} - \tfrac{1}{2}[T^a, T^b], \\ i f^{abc} T^c &= [T^a, T^b],\end{aligned} \tag{S.44}$$

so indeed, every term in eq. (S.36) can be written in the form (6). Specifically,

$$\begin{aligned}F &= -\frac{2g^2 (e_1^* p)(e_2^* p')}{t-m^2} - \frac{2g^2 (e_1^* p')(e_2^* p)}{u-m^2} - g^2 (e_1^* e_2^*), \\ iG &= +\frac{2g^2 (e_1^* p)(e_2^* p')}{t-m^2} - \frac{2g^2 (e_1^* p')(e_2^* p)}{u-m^2} \\ & \quad - \frac{g^2}{s}\Big[(u-t)(e_1^* e_2^*) + 2(e_1^* k_2)\big(e_2^*(p-p')\big) - 2(e_2^* k_1)\big(e_1^*(p-p')\big)\Big].\end{aligned} \tag{S.45}$$

Problem **4**(b):

First, let us average over the scalar particles' color indices $i,j=1,2,\ldots,\dim(r)$. For fixed gauge bosons a and b, let

$$M \;=\; F\{T^a_{(r)},T^b_{(r)}\} \;+\; iG\,[T^a_{(r)},T^b_{(r)}] \tag{S.46}$$

be a matrix (in the representation (r) of the gauge group) whose elements M^j_i are annihilation amplitudes (6) for the scalar particles Φ_i and Φ^{*j} of specific colors i,j. Then averaging over those colors gives

$$\frac{1}{\dim^2(r)}\sum_{i,j}\left|M^j_i\right|^2 \;=\; \frac{1}{\dim^2(r)}\sum_{i,j}M^j_i(M^\dagger)^i{}_j = \frac{1}{\dim^2(r)}\,\mathrm{tr}(MM^\dagger). \tag{S.47}$$

For the specific form (S.46) of the matrix M, we write

$$\begin{aligned} M \;&=\; (F+iG)\,T^a_{(r)}T^b_{(r)} \;+\; (F-iG)\,T^b_{(r)}T^a_{(r)}\,,\\ M^\dagger \;&=\; (F+iG)^*\,T^b_{(r)}T^a_{(r)} \;+\; (F-iG)^*\,T^a_{(r)}T^b_{(r)}\,, \end{aligned} \tag{S.48}$$

and therefore

$$\begin{aligned} \mathrm{tr}(MM^\dagger) \;&=\; |F+iG|^2\;\mathrm{tr}_{(r)}(T^aT^bT^bT^a) \;+\; (F-iG)(F+iG)^*\;\mathrm{tr}_{(r)}(T^bT^aT^bT^a)\\ &\qquad +\; (F+iG)(F-iG)^*\;\mathrm{tr}_{(r)}(T^aT^bT^aT^b) \;+\; |F-iG|^2\;\mathrm{tr}_{(r)}(T^bT^aT^aT^b)\\ &=\; 2(|F|^2+|G|^2)\;\mathrm{tr}_{(r)}(T^aT^aT^bT^b) \;+\; 2(|F|^2-|G|^2)\;\mathrm{tr}_{(r)}(T^aT^bT^aT^b)\\ &=\; 4|F|^2\;\mathrm{tr}_{(r)}(T^aT^aT^bT^b) \;+\; 2(|G|^2-|F|^2)\,\mathrm{tr}_{(r)}(T^a[T^a,T^b]T^b). \end{aligned} \tag{S.49}$$

where the second equality follows from the cyclic symmetry of the traces.

Our next step is to sum over the color indices a and b of the gauge bosons. In the context of eq. (S.49), we have

$$\sum_{a,b}\mathrm{tr}_{(r)}(T^aT^aT^bT^b) \;=\; \mathrm{tr}_{(r)}\left(\left(\sum_a T^aT^a\right)\left(\sum_b T^bT^b\right)\right) \;=\; \mathrm{tr}_{(r)}\,(C_2\,C_2) \;=\; C^2(r)\times\dim(r) \tag{S.50}$$

and

$$
\begin{aligned}
\sum_{a,b} \operatorname{tr}_{(r)}(T^a[T^a,T^b]T^b) &= \sum_{a,b}\sum_{c} if^{abc}\operatorname{tr}_{(r)}(T^aT^cT^b) \\
&= \tfrac{1}{2}\sum_{a,b,c} if^{abc}\operatorname{tr}_{(r)}(T^aT^cT^b - T^aT^bT^c) \\
&= \tfrac{1}{2}\sum_{a,b,c} if^{abc}\sum_{d} if^{cbd}\operatorname{tr}_{(r)}(T^aT^d) \\
&= \tfrac{1}{2}\sum_{a,b,c,d} (if^{abc})(if^{dbc})\times R(r)\delta^{ad} \\
&= \tfrac{1}{2}R(r)\sum_{a}\sum_{b,c}\Big(if^{abc} = (T^a_{(\mathrm{adj})})^{bc}\Big)\Big(if^{acb} = (T^a_{(\mathrm{adj})})^{cb}\Big) \\
&= \tfrac{1}{2}R(r)\sum_{a}\operatorname{tr}\Big(T^a_{(\mathrm{adj})}T^a_{(\mathrm{adj})}\Big) \\
&= \tfrac{1}{2}R(r)\times C(G)\dim(G) \\
&= \tfrac{1}{2}C(G)C(r)\dim(r).
\end{aligned}
\tag{S.51}
$$

Therefore,

$$
\sum_{a,b}\operatorname{tr}(MM^\dagger) \;=\; C(r)\dim(r)\times\big[4C(r)|F|^2 \;+\; C(G)\big(|G|^2-|F|^2\big)\big] \tag{S.52}
$$

and hence in light of eq. (S.47),

$$
\frac{1}{\dim^2(r)}\sum_{ij}\sum_{ab}|\mathcal{M}|^2 \;=\; \frac{C(r)}{\dim(r)}\times\Big(4C(r)\,|F|^2 \,+\, C(\mathrm{adj})\,(|G|^2-|F|^2)\Big)\,. \tag{9}
$$

Eq. (10) follows from this as a special case.

Problem **4**(c):

Let us take a closer look at eqs. (S.45). In the center of mass frame, $\mathbf{p}' = -\mathbf{p}$, $\mathbf{k}_2 = -\mathbf{k}_1$, and the vector bosons' polarizations $e^\mu_{1,2}$ are purely spatial and transverse, $e^0_{1,2} = 0$ and $\mathbf{k}_{1,2}\cdot\mathbf{e}_{1,2} = 0$. Consequently, eqs. (S.45) simplify to

$$
\begin{aligned}
F &= 2g^2(\mathbf{e}_1^*\mathbf{p})(\mathbf{e}_2^*\mathbf{p})\left(\frac{1}{t-m^2} + \frac{1}{u-m^2}\right) \;+\; g^2(\mathbf{e}_1^*\mathbf{e}_2^*), \\
G &= 2ig^2(\mathbf{e}_1^*\mathbf{p})(\mathbf{e}_2^*\mathbf{p})\left(\frac{1}{t-m^2} - \frac{1}{u-m^2}\right) \;-\; ig^2\,\frac{u-t}{s}\,(\mathbf{e}_1^*\mathbf{e}_2^*).
\end{aligned}
\tag{S.53}
$$

Furthermore, in the center of mass frame $E = E' = \omega_1 = \omega_2$, $|\mathbf{k}| = \omega = E$, $|\mathbf{p}| = \beta E$,

$$s \;=\; 4E^2, \qquad t - m^2 \;=\; -2E^2(1 - \beta\cos\theta), \qquad u - m^2 \;=\; -2E^2(1 + \beta\cos\theta),$$

hence

$$\begin{aligned} \frac{u-t}{s} &= \beta\cos\theta, \\ \frac{1}{t-m^2} + \frac{1}{u-m^2} &= \frac{-1}{m^2 + \mathbf{p}^2\sin^2\theta}, \\ \frac{1}{t-m^2} - \frac{1}{u-m^2} &= \frac{-\beta\cos\theta}{m^2 + \mathbf{p}^2\sin^2\theta}, \end{aligned}$$

and therefore

$$\begin{aligned} F &= g^2\left((\mathbf{e}_1^*\mathbf{e}_2^*) - \frac{2(\mathbf{e}_1^*\mathbf{p})(\mathbf{e}_2^*\mathbf{p})}{m^2 + \mathbf{p}^2\sin^2\theta}\right), \\ G &= -ig^2\left((\mathbf{e}_1^*\mathbf{e}_2^*) + \frac{2(\mathbf{e}_1^*\mathbf{p})(\mathbf{e}_2^*\mathbf{p})}{m^2 + \mathbf{p}^2\sin^2\theta}\right) \times \beta\cos\theta. \end{aligned} \tag{S.54}$$

Now consider the gluons' polarization vectors. For the problem at hand it is easier to use linear polarizations for which the $\mathbf{e}_1$ and $\mathbf{e}_2$ are real unit vectors. Specifically, for each gluon there is a choice of two transverse $\mathbf{e}$, one parallel to the $(\mathbf{p}, \mathbf{k})$ plane and one perpendicular to it. In the coordinate system where

$$\mathbf{p} \;=\; \beta E(0, 0, 1) \qquad \text{and} \quad \mathbf{k} \;=\; E(\sin\theta, 0, \cos\theta), \tag{S.55}$$

the two polarization vectors are

$$\mathbf{e}_\parallel \;=\; (-\cos\theta, 0, +\sin\theta) \quad \text{and} \quad \mathbf{e}_\perp \;=\; (0, 1, 0). \tag{S.56}$$

For these vectors

$$(\mathbf{p}\mathbf{e}_\parallel) \;=\; \beta E\sin\theta, \qquad (\mathbf{p}\mathbf{e}_\perp) \;=\; 0, \tag{S.57}$$

so according to eqs. (S.54),

$$\text{for } \mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_\perp\,, \quad F \;=\; g^2 \quad \text{and} \quad G \;=\; -ig^2\beta\cos\theta, \tag{S.58}$$

$$\text{for } \mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_{\|}\,, \quad F \;=\; g^2(1-2A) \quad \text{and} \quad G \;=\; -ig^2(1+2A)\beta\cos\theta \tag{S.59}$$

where

$$A \;=\; \frac{\mathbf{p}^2\sin^2\theta}{m^2+\mathbf{p}^2\sin^2\theta}\,, \tag{S.60}$$

and finally

$$\text{for } \mathbf{e}_1 = \mathbf{e}_\perp,\ \mathbf{e}_2 = \mathbf{e}_{\|} \text{ or } \textit{vice verse}, \quad F \;=\; G \;=0. \tag{S.61}$$

Problem **4**(d):

According to eq. (S.61), the two gauge bosons produced in the $\Phi\Phi^*$ annihilation must have similar polarizations: either both are polarized $\|$ to the $(\mathbf{p},\mathbf{k})$ plane of scattering or both are polarized $\perp$ to the plane. Consequently, there are only two polarized partial cross sections to consider, namely

$$\begin{aligned}\left(\frac{d\sigma(\perp)}{d\Omega}\right)_{\text{c.m.}} &= \frac{g^4}{64\pi^2E^2_{\text{c.m.}}\beta}\,\frac{C(r)}{\dim(r)}\Bigl(4C(r)\;-\;(1-\beta^2\cos^2\theta)C(G)\Bigr),\\ \left(\frac{d\sigma(\|)}{d\Omega}\right)_{\text{c.m.}} &= \frac{g^4}{64\pi^2E^2_{\text{c.m.}}\beta}\,\frac{C(r)}{\dim(r)}\Bigl(4C(r)(1-2A)^2\;-\;C(G)\bigl((1-2A)^2-\beta^2(1+2A)^2\cos^2\theta\bigr)\Bigr).\end{aligned} \tag{S.62}$$

Note that the angular dependence of the $\|$ polarized partial cross section is more complicated than it looks because A is θ–dependent according to eq. (S.60).

In the limit of non-relativistic scalar particles, $\beta \ll 1$ leads to $A \ll 1$ and hence to the expected isotropy and polarization independence of the annihilation cross-section,

$$\left(\frac{d\sigma(\|)}{d\Omega}\right)_{\text{c.m.}} \;\approx\; \left(\frac{d\sigma(\perp)}{d\Omega}\right)_{\text{c.m.}} \;\approx\; \frac{g^4}{256\pi^2m^2\beta}\,\frac{C(r)\bigl(4C(r)-C(G)\bigr)}{\dim(r)}\,. \tag{S.63}$$

In the opposite limit of ultra-relativistic scalars, $\beta\approx 1$ leads to $A\approx 1$ (except for $\theta\approx 0$) and therefore

$$\begin{aligned}\left(\frac{d\sigma(\perp)}{d\Omega}\right)_{\text{c.m.}} &\approx \frac{g^4}{16\pi^2E^2_{\text{c.m.}}}\,\frac{C^2(r)}{\dim(r)}\,,\\ \left(\frac{d\sigma(\|)}{d\Omega}\right)_{\text{c.m.}} &\approx \frac{g^4}{16\pi^2E^2_{\text{c.m.}}}\,\frac{C(r)}{\dim(r)}\left[C(r)\;+\;C(G)\,\frac{9\cos^2\theta-1}{4}\right].\end{aligned} \tag{S.64}$$