Electric Form Factor

Now consider the electric form factor $F_1(q^2)$. Let's start by calculating the momentum integral in eq. (30). This time, the numerator $\mathcal{N}_1$ depends on ℓ as $a\ell^2 + b$ and there is a logarithmic divergence for $\ell \to \infty$; to regularize this divergence, we work in $D = 4 - 2\epsilon$ dimensions. Thus,

$$\begin{aligned}
-2i\int\limits_{\text{reg}}\frac{d^4\ell}{(2\pi)^4}\,\frac{a\ell^2+b}{[\ell^2-\Delta+i0]^3} \;&\equiv\; -2i\mu^{4-D}\int\frac{d^D\ell}{(2\pi)^D}\,\frac{a\ell^2+b}{[\ell^2-\Delta+i0]^3} \;= \\
&= \; -2i\mu^{4-D}\int\frac{id^D\ell_E}{(2\pi)^D}\,\frac{-a\ell_E^2+b}{-[\ell_E^2+\Delta]^3} \\
&= \; \mu^{4-D}\int\frac{d^D\ell_E}{(2\pi)^D}\,(a\ell_E^2-b)\times\left[\frac{2}{[\ell_E^2+\Delta]^3} \;=\int\limits_0^\infty dt\,t^2\,e^{-t(\ell_E^2+\Delta)}\right] \\
&= \int\limits_0^\infty dt\,t^2 e^{-t\Delta}\times\mu^{4-D}\int\frac{d^D\ell_E}{(2\pi)^D}\left[(a\ell_E^2-b)e^{-t\ell_E^2} = \left(-a\frac{\partial}{\partial t}-b\right)e^{-t\ell_E^2}\right] \\
&= \int\limits_0^\infty dt\,t^2 e^{-t\Delta}\left(-a\frac{\partial}{\partial t}-b\right)\frac{\mu^{4-D}}{(4\pi\,t)^{D/2}} \\
&= \; \frac{\mu^{4-D}}{(4\pi)^{D/2}}\int\limits_0^\infty dt\,e^{-t\Delta}\times\left(a\times\tfrac{D}{2}t^{1-(D/2)} \;-\; b\times t^{2-(D/2)}\right) \\
&= \; \frac{\mu^{4-D}}{(4\pi)^{D/2}}\left\{a\times\tfrac{D}{2}\Gamma\left(2-\tfrac{D}{2}\right)\times\Delta^{\frac{D}{2}-2} \;-\; b\times\Gamma\left(3-\tfrac{D}{2}\right)\times\Delta^{\frac{D}{2}-3}\right\} \\
&\to \; \frac{(4\pi\mu)^\epsilon}{16\pi^2}\times\frac{\Gamma(1+\epsilon)}{\Delta^\epsilon}\times\left\{\frac{2-\epsilon}{\epsilon}\times a \;-\; \frac{b}{\Delta}\right\}.
\end{aligned} \tag{47}$$

In light of eq. (27),

$$a \;=\; \frac{(D-2)^2}{D}\,, \qquad b \;=\; 2z\times(2m^2-q^2) \;-\; (D-2)\times\Delta, \tag{48}$$

so on the last line of eq. (47)

$$\frac{2-\epsilon}{\epsilon}\times a - \frac{b}{\Delta} \;=\; \frac{2-\epsilon}{\epsilon}\times\frac{2(1-\epsilon)^2}{2-\epsilon} - \frac{2z(2m^2-q^2)}{\Delta} + (2-2\epsilon) \;=\; \frac{2(1-\epsilon)}{\epsilon} - \frac{2z(2m^2-q^2)}{\Delta}\,. \tag{49}$$

Consequently, the momentum integral in eq. (30) for the electric form factors evaluates to

$$-2ie^2\mu^{4-D}\int\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_1}{[\ell^2-\Delta+i0]^3}\ = \qquad (50)$$

$$=\ \frac{\alpha}{4\pi}\left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon}\left\{\Gamma(\epsilon)\times 2(1-\epsilon)\ -\ \Gamma(1+\epsilon)\times\frac{2z\times(2m^2-q^2)}{\Delta}\right\},$$

and now we need to integrate this expression over the Feynman parameters.

Changing the integration variables from x, y, z to w and ξ according to eq. (44), we have

$$F_1^{1\,\mathrm{loop}}(q^2)\ =\ \frac{\alpha}{4\pi}\,(4\pi\mu^2)^{\epsilon}\int_0^1 d\xi\int_0^1 dw\,w\times\left\{\begin{aligned}&2(1-\epsilon)\Gamma(\epsilon)\times\frac{1}{[\Delta(w,\xi)]^{\epsilon}}\\ &-2\Gamma(1+\epsilon)\times\frac{(1-w)(2m^2-q^2)}{[\Delta(w,\xi)]^{1+\epsilon}}\end{aligned}\right\} \qquad (51)$$

where

$$\Delta(w,\xi)\ =\ (1-z)^2m^2\ -\ xyq^2\ =\ w^2\times\Bigl(m^2\ -\ \xi(1-\xi)q^2\Bigr), \qquad (52)$$

or equivalently,

$$\Delta(w,\xi)\ =\ w^2\times H(\xi)\quad\text{where}\quad H(\xi)\ \stackrel{\text{def}}{=}\ m^2\ -\ \xi(1-\xi)q^2. \qquad (53)$$

The form (53) is particularly convenient for evaluating the $\int dw$ integral in eq. (51), which becomes

$$\int_0^1 dw\left\{\frac{2(1-\epsilon)\Gamma(\epsilon)}{H^{\epsilon}}\times\frac{w}{w^{2\epsilon}}\ -\ 2\Gamma(1+\epsilon)\times\frac{2m^2-q^2}{H^{1+\epsilon}}\times\frac{w(1-w)}{w^{2+2\epsilon}}\right\}. \qquad (54)$$

Near the lower limit $w\to 0$, the integrand is dominated by the second term, which is proportional to $w^{-1-2\epsilon}$. But for any $\epsilon\geq 0$ — *i.e.*, for any dimension $D\leq 4$ — the integral

$$\int_0^{\text{positive}} dw\,\frac{1}{w^{1+2\epsilon}} \qquad (55)$$

diverges: For $D=4$ this divergence is logarithmic while for $D<4$ it becomes power-like.

Physically, this divergence is *infrared* rather than ultraviolet, that's why it gets worse as we lower the dimension D. Indeed, let's go back to the diagram (1) and look at the denominator $\mathcal{D}$ in eqs. (2) and (4). Taking the electron's momenta p and p' on-shell before introducing the Feynman parameters, we have

$$(p+k)^2 - m^2 = k^2 + 2kp \quad \text{and likewise} \quad (p'+k)^2 - m^2 = k^2 + 2kp'. \tag{56}$$

Consequently, for $k \to 0$, the denominator behaves as $\mathcal{D} \propto k^4$ while the numerator $\mathcal{N}^\mu$ remains finite, and the integral

$$\int d^D k \, \frac{\mathcal{N}^\mu}{\mathcal{D}} \propto \int d^D k \, \frac{1}{k^4} \tag{57}$$

diverges for $k \to 0$. In $D = 4$ dimensions, the infrared divergence here is logarithmic, while in lower dimensions $D < 4$ it becomes power-like, *i.e.* $O\left((1/k_{\text{min}})^{4-D}\right)$ — precisely as in eqs. (55) and (54).

We can regularize the infrared divergence (57) — and also (55) — by analytically continuing spacetime dimension to $D > 4$. Such dimensional regularization of an IR divergence is used in many situations in both QFT and condensed matter. However, taking $D > 4$ makes the ultraviolet divergences worse, so if some amplitude has both UV and IR divergences, we cannot cure both of them at the same time by analytically continuing to $D \neq 4$. In particular, when calculating the electric form factor $F_1(q^2)$ of the electron, we need $D < 4$ to regulate the momentum integral $\int d^D\ell$, but then we need $D > 4$ to regulate the integral over the Feynman parameters.

A common *dirty trick* is to first continue to $D < 4$ and evaluate the $\int d^D\ell$ momentum integral, then analytically continue the result to $D > 4$ and integrate over the Feynman parameters, and then continue the final result to $D = 4$. However, in this kind of dimensional regularization its hard to disentangle the $1/\epsilon$ poles coming from the UV divergence $\log(\Lambda^2/\mu^2)$ from the $1/\epsilon$ poles coming from the IR divergence $\log(\mu^2/k^2_{\text{min}})$, so we are not going to use it here.

Instead, we are going to use DR for the UV divergence only and regulate the IR divergence by giving the photon a tiny mass $m_\gamma^2 \ll m_e^2$. Strictly speaking, a massive vector particle has three polarization states and its propagator is

$$= \frac{-i}{k^2 - m_\gamma^2 + i0} \times \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m_\gamma^2} \right). \tag{58}$$

However, the longitudinal polarization of the massive but ultra-relativistic photon does not couple to a conserved current, so we are going to disregard the $k^\mu k^\nu$ terms in the propagator (58) and use

$$= \frac{-ig^{\mu\nu}}{k^2 - m_\gamma^2 + i0}. \tag{59}$$

In other words, we use the Feynman gauge despite of the photon's mass; this is not completely consistent, but the inconsistencies go away in the $m_\gamma \to 0$ limit.

Using this infrared regulator for the internal photon line in the one-loop diagram (1), we get the vertex amplitude that looks exactly like eq. (2) except for one factor in the denominator,

$$\frac{1}{k^2 + i0} \quad \text{becomes} \quad \frac{1}{k^2 - m_\gamma^2 + i0}. \tag{60}$$

In terms of the integral (12), this change has no effect on the numerator $\mathcal{N}^\mu$ or the loop momentum ℓ (which remains exactly as in eq. (7)), but the Δ in the denominator becomes

$$\Delta'(x, y, z) = \Delta(x, y, z) + z \times m_\gamma^2. \tag{61}$$

Consequently, the electric form factor is

$$F_1^{1\,\text{loop}}(q^2) = \int d(FP) \int \mu^{4-D} \frac{d^D\ell}{(2\pi)^D} \frac{-2ie^2 \times \mathcal{N}_1}{[\ell^2 - \Delta' + i0]^3}, \tag{62}$$

exactly as in eq. (30) except for Δ' instead of Δ in the denominator. The momentum integral here converges for any $D < 4$ and it evaluates exactly as in eq. (47). The only subtlety here is that in the numerator, the ℓ–independent term b involves the un-modified Δ instead of Δ' (*cf.* eq. (48)), but we can fix that by writing

$$b \;=\; 2z\times\left(2m_e^2-q^2+(1-\epsilon)m_\gamma^2\right)\;-\;2(1-\epsilon)\times\Delta'. \tag{63}$$

Hence, instead of eq. (51), we get

$$F_1^{1\,\mathrm{loop}}(q^2) \;=\; \frac{\alpha}{4\pi}\,(4\pi\mu^2)^\epsilon\int_0^1\!d\xi\int_0^1\!dw\,w\times\left\{\begin{aligned}&2(1-\epsilon)\Gamma(\epsilon)\times\frac{1}{[\Delta'(w,\xi)]^\epsilon}\\&-2\Gamma(1+\epsilon)\times\frac{(1-w)(2m_e^2-q^2+(1-\epsilon)m_\gamma^2)}{[\Delta'(w,\xi)]^{1+\epsilon}}\end{aligned}\right\} \tag{64}$$

where

$$\Delta'(w,\xi) \;=\; (1-z)^2m_e^2\;-\;xyq^2\;+\;zm_\gamma^2\;=\;w^2\times H(\xi)\;+\;(1-w)\times m_\gamma^2\,. \tag{65}$$

Note that the photon's mass is tiny, $m_\gamma^2\ll m_e^2,q^2$; were it not for the IR divergences, we would have used $m_\gamma^2=0$. This allows us to neglect various $O(m_\gamma^2)$ terms in eq. (64) except when it would cause a divergence for $w\to0$; in particular, we may neglect the $(1-\epsilon)m_\gamma^2$ term in the numerator of the second term in the integrand. As to the denominators, in eq. (65) the second term containing the photon's mass becomes important only in the $w\to0$ limit, and in that limit $(1-w)m_\gamma^2\to m_\gamma^2$. Thus, we approximate

$$\Delta'(w,\xi)\;\approx\;w^2\times H(\xi)\;+\;m_\gamma^2 \tag{66}$$

and the $\int dw$ integral in eq. (64) becomes

$$\int_0^1 dw\, w \times \left\{ 2(1-\epsilon)\Gamma(\epsilon) \times \frac{1}{[w^2 H(\xi) + m_\gamma^2]^\epsilon} \;-\; 2\Gamma(1+\epsilon) \times \frac{(1-w)(2m_e^2 - q^2)}{[w^2 H(\xi) + m_\gamma^2]^{1+\epsilon}} \right\}$$

$$\begin{aligned} &= \frac{2(1-\epsilon)\Gamma(\epsilon)}{H^\epsilon} \times \int_0^1 \frac{dw\, w}{[w^2 + (m_\gamma^2/H)]^\epsilon} \\ &\quad + 2\Gamma(1+\epsilon)\, \frac{2m_e^2 - q^2}{H^{1+\epsilon}} \times \int_0^1 \frac{dw\, w^2}{[w^2 + (m_\gamma^2/H)]^{1+\epsilon}} \\ &\quad - 2\Gamma(1+\epsilon)\, \frac{2m_e^2 - q^2}{H^{1+\epsilon}} \times \int_0^1 \frac{dw\, w}{[w^2 + (m_\gamma^2/H)]^{1+\epsilon}} \,. \end{aligned} \qquad (67)$$

For $0 < \epsilon < \frac{1}{2}$ — *i.e.*, for $3 < D < 4$ — the integrals on the second and third lines here converge even for $m_\gamma^2 = 0$,

$$\begin{aligned} \int_0^1 \frac{dw\, w}{[w^2]^\epsilon} &= \frac{1}{2 - 2\epsilon} \quad \text{for } \epsilon < 1, \\ \int_0^1 \frac{dw\, w^2}{[w^2]^{1+\epsilon}} &= \frac{1}{1 - 2\epsilon} \quad \text{for } \epsilon < \tfrac{1}{2}, \end{aligned} \qquad (68)$$

so we may just as well evaluate them without the photon's mass. Only on the last line of eq. (67) we do need $m_\gamma^2 \neq 0$ to make the integral converge for some $D \leq 4$:

$$\int_0^1 \frac{dw\, w}{[w^2 + (m_\gamma^2/H)]^{1+\epsilon}} \;=\; \frac{-1}{2\epsilon}\, \frac{1}{[w^2 + (m_\gamma^2/H)]^\epsilon}\bigg|_0^1 \;=\; \frac{1}{2\epsilon}\left[\left(\frac{H}{m_\gamma^2} \right)^\epsilon - 1 \right]. \qquad (69)$$

Combining all these $\int dw$ integrals together, we get

$$\int_0^1 dw\Big\{\cdots\Big\} = \frac{\Gamma(\epsilon)}{H^\epsilon} + \frac{2\Gamma(1+\epsilon)}{1-2\epsilon}\times\frac{2m_e^2-q^2}{H^{1+\epsilon}} - \frac{\Gamma(1+\epsilon)}{\epsilon}\times\frac{2m_e^2-q^2}{H^{1+\epsilon}}\times\left[\left(\frac{H}{m_\gamma^2}\right)^\epsilon - 1\right]$$
$$= \frac{\Gamma(\epsilon)}{H^\epsilon}\times\left\{1 + \frac{2m_e^2-q^2}{H}\times\left[\frac{1}{1-2\epsilon} - \left(\frac{H}{m_\gamma^2}\right)^\epsilon\right]\right\} \tag{70}$$

and hence

$$F_1^{1\,\text{loop}}(q^2) = \frac{\alpha}{4\pi}\int_0^1 d\xi\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{H(\xi)}\right)^\epsilon\times\left\{1 + \frac{2m_e^2-q^2}{H(\xi)}\times\left[\frac{1}{1-2\epsilon} - \left(\frac{H(\xi)}{m_\gamma^2}\right)^\epsilon\right]\right\} \tag{71}$$

where

$$H(\xi) = m_e^2 - \xi(1-\xi)q^2. \tag{53}$$

Before we even try to perform this last integral, let's remember that

$$\Gamma^\mu_{\text{net}} = \gamma^\mu_{\text{tree}} + \Gamma^\mu\text{loops} + \delta_1\times\gamma^\mu \tag{72}$$

and hence

$$F_1^{\text{net}}(q^2) = 1^{\text{tree}} + F_1^{\text{loops}}(q^2) + \delta_1\,. \tag{73}$$

Also, there is no renormalization of the net charge, so

$$F_1^{\text{net}}(q^2) \to 1 \quad \text{for } q^2\to 0 \tag{74}$$

and hence

$$\delta_1 = -F_1^{\text{loops}}(q^2=0). \tag{75}$$

To calculate the counterterm δ_1 to order α we use eq. (71) for $q^2=0$, in which case $H(\xi)\equiv m_e^2$

and the $\int d\xi$ becomes trivial (the integrand does not depend on ξ at all). Thus,

$$\delta_1 \;=\; -\frac{\alpha}{4\pi}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^{\epsilon}\times\left\{1\,+\,\frac{2}{1-2\epsilon}\,-\,2\left(\frac{m_e^2}{m_\gamma^2}\right)^{\epsilon}\right\}\,+\,O(\alpha^2). \tag{76}$$

This formula holds for any dimension D between 3 and 4 (*i.e.*, $0<\epsilon<\frac{1}{2}$). In the $D\to 4$ limit, it becomes

$$\delta_1 \;=\; -\frac{\alpha}{4\pi}\times\left\{\frac{1}{\epsilon}\,-\,\gamma_E\,+\,\log\frac{4\pi\mu^2}{m_e^2}\,+\,4\,-\,2\log\frac{m_e^2}{m_\gamma^2}\right\}+\,O(\alpha^2). \tag{77}$$

Now let's go back to the electric form factor $F_1^{\rm net}(q^2)$ for $q^2\neq 0$. According to eqs. (73) and (75), at the one-loop level

$$F_1^{\rm net}(q^2)\,-\,1\;=\;F_1^{\rm 1\,loop}(q^2)\,-\,F_1^{\rm 1\,loop}(0)\,+\,O(\alpha^2) \tag{78}$$

where $F_1^{\rm 1\,loop}(q^2)$ is given by eq. (71). Taking the $\epsilon\to 0$ limit of that formula, we arrive at

$$F_1^{\rm 1\,loop}(q^2)\;=\;\frac{\alpha}{4\pi}\int_0^1\!d\xi\,\left\{\frac{1}{\epsilon}\,-\,\gamma_E\,+\,\log\frac{4\pi\mu^2}{H(\xi)}\,+\,\frac{2m_e^2-q^2}{H(\xi)}\times\left[2\,-\,\log\frac{H(\xi)}{m_\gamma^2}\right]\right\}, \tag{79}$$

and now we should subtract a similar a similar expression for $q^2=0$. This subtraction cancels the UV divergence and the associated $1/\epsilon$ pole but not the IR divergence. Moreover, not only the subtracted one-loop amplitude depends on the IR regulators, but the coefficient of the $\log m_\gamma^2$ has a non-trivial momentum dependence. Indeed,

$$\begin{aligned} F_1^{\rm 1\,loop}(q^2)\,-\,F_1^{\rm 1\,loop}(0) & \\ &=\;\frac{\alpha}{4\pi}\int_0^1\!d\xi\,\left\{\log\frac{m_e^2}{H(\xi)}\,+\,\frac{2m_e^2-q^2}{H(\xi)}\times\left[2\,-\,\log\frac{H(\xi)}{m_\gamma^2}\right]\,-\,2\left[2\,-\,\log\frac{m_e^2}{m_\gamma^2}\right]\right\}\\ &=\;\frac{\alpha}{4\pi}\times\left\{-f_{\rm IR}(q^2/m_e^2)\times\log\frac{O(m_e^2\text{ or }q^2)}{m_\gamma^2}\,+\,\text{a\_finite\_function}(q^2/m_e^2)\right\} \end{aligned} \tag{80}$$

where 'a finite function' means a function of q^2/m_e^2 which remains finite when we remove the IR regulator and set the photon's mass to zero, and

$$f_{\mathrm{IR}}(q^2/m_e^2) \;=\; \int_0^1 d\xi \left(\frac{2m_e^2 - q^2}{H(\xi)} \;-\; 2 \;=\; \frac{-q^2 \times (1 - 2\xi + 2\xi^2)}{m_e^2 - q^2 \times \xi(1-\xi)} \right) \tag{81}$$

is the same function that governs the soft-photon bremsstrahlung. In terms of §6.1 of the Peskin & Schroeder textbook,

$$f_{\mathrm{IR}}(q^2/m_e^2) \;=\; \mathcal{I}(\mathbf{v}, \mathbf{v}') \;=\; \int \frac{d^2\Omega_{\mathbf{n}}}{4\pi} \left[-\left(\frac{p'^\mu}{(np')} \;-\; \frac{p^\mu}{(np)} \right)^2 \right]^{n^0=|\mathbf{n}|=1}, \tag{82}$$

see textbooks eqs. (6.69–70) for the proof.
Note: my definition of the F_{IR} differs from the textbook's by a factor of 2.

Altogether, the electric form factor of the electron is

$$F_1^{\mathrm{net}}(q^2) \;=\; 1 \;-\; \frac{\alpha}{4\pi} \times \left\{ f_{\mathrm{IR}}(q^2/m_e^2) \times \log \frac{O(m_e^2 \text{ or } q^2)}{m_\gamma^2} \;+\; \mathrm{finite}(q^2/m_e^2) \right\} \;+\; O(\alpha^2). \tag{83}$$

Implications of this formula will be discussed in class; see also §6.4 of the textbook.