Konishi Anomaly

Consider the SQED with massless charged fields A and B. Classically, it has an axial symmetry $A \to e^{+i\varphi}A$, $B \to e^{+i\varphi}B$ and hence a conserved axial current

$$J_{\rm ax} \;=\; \overline{A}\,e^{+2gV}A \;+\; \overline{B}\,e^{-2gV}B, \tag{1}$$

$$D^2 J_{\rm ax} \;=\; \overline{D}^2 J_{\rm ax} \;=\; 0 \quad \text{(classically)}. \tag{2}$$

In components

$$\begin{aligned} J_{\rm ax}(x,\theta,\bar\theta) \;&=\; (\theta\sigma^\mu\bar\theta)\times j^{5\mu}(x) \;+\; \text{other combinations of } \theta \text{ and } \bar\theta, & (3)\\ j^{5\mu} \;&=\; \overline{\psi}_A\bar\sigma^\mu\psi_A \;+\; \overline{\psi}_B\bar\sigma^\mu\psi_B \;+\; (iA^\dagger\mathcal{D}^\mu A - iA\mathcal{D}^\mu A^\dagger) \;+\; (iB^\dagger\mathcal{D}^\mu B - iB\mathcal{D}^\mu B^\dagger) \\ &=\; \overline{\Psi}\gamma^5\gamma^\mu\Psi_{\rm Dirac} \;+\; \text{bosonic}. & (4) \end{aligned}$$

Eqs. (2) imply *inter alia* that the ordinary axial current $j^{5\mu}(x)$ is conserved, $\partial_\mu j^{5\mu} = 0$.

In the ordinary QED with a massless electron, the loop corrections destroy the conservation of the axial current. Instead of $\partial_\mu j^{5\mu} = 0$, we have the Adler–Bell–Jackiw anomaly

$$\partial_\mu j^{5\mu} \;=\; \frac{e^2}{32\pi^2}\,\epsilon^{\kappa\lambda\mu\nu}F_{\kappa\lambda}F_{\mu\nu}\,. \tag{5}$$

The superfield analogue of this anomaly is the Konishi anomaly: instead of the classical eqs. (2), the current superfield $J_{\rm ax}$ satisfies

$$\begin{aligned} -\tfrac14\overline{D}^2 J_{\rm ax} \;&=\; \frac{g^2}{8\pi^2}\,W^\alpha W_\alpha\,, \\ -\tfrac14 D^2 J_{\rm ax} \;&=\; \frac{g^2}{8\pi^2}\,\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\,. \end{aligned} \tag{6}$$

Before we derive the Konishi anomaly, let's verify that it includes the ordinary Adler–Bell–Jackiw anomaly (5). Reversing eq. (3), we have

$$j^{5\mu}(x) \;=\; \tfrac14\sigma^\mu_{\alpha\dot\alpha}\,\left[\overline{D}^{\dot\alpha}, D^\alpha\right] J_{\rm ax}(x,\theta,\bar\theta)\Big|_{\theta=\bar\theta=0} \tag{7}$$

and consequently

$$\partial_\mu j^{5\mu}(x) \;=\; \tfrac{1}{4}\partial_{\alpha\dot\alpha}\,\left[\overline{D}^{\dot\alpha}, D^\alpha\right] J_{\rm ax}(x,\theta,\bar\theta)\Big|_{\theta=\bar\theta=0} \;=\; \frac{i}{16}\,\left[\overline{D}^2, D^2\right] J_{\rm ax}(x,\theta,\bar\theta)\Big|_{\theta=\bar\theta=0} . \tag{8}$$

The Konishi anomaly (6) gives us

$$\left[\overline{D}^2, D^2\right] J_{\rm ax} \;=\; \frac{g^2}{2\pi^2} \times \Big(D^2(W^\alpha W_\alpha) \,-\, \overline{D}^2(\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}) \Big) , \tag{9}$$

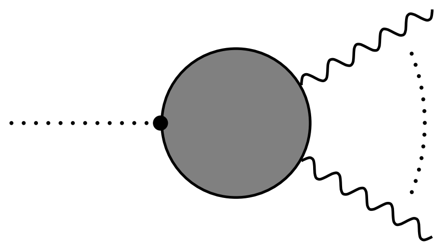
hence in the context of eq. (8),

$$\begin{aligned}
\partial_\mu j^{5\mu} \;&=\; \frac{ig^2}{32\pi^2}\left[D^2(W^\alpha W_\alpha) \,-\, \overline{D}^2(\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha})\right]_{\theta=\bar\theta=0} \\
&=\; \frac{-ig^2}{8\pi^2}\int d^2\theta\, W^\alpha W_\alpha \;+\; \frac{ig^2}{8\pi^2}\int d^2\bar\theta\, \overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha} \\
&=\; \frac{g^2}{4\pi^2}\times {\rm Im}\int d^2\theta\, W^\alpha W_\alpha \\
&=\; \frac{g^2}{16\pi^2}\times \epsilon^{\kappa\lambda\mu\nu} F_{\kappa\lambda}F_{\mu\nu} ,
\end{aligned} \tag{10}$$

Since in our normalization $g^2 = \frac{1}{2}e^2$, this is precisely the Adler–Bell–Jackiw anomaly (5).

Conversely, given the Adler–Bell–Jackiw anomaly and SUSY, applying supersymmetry transformations to both sides of eq. (5) gives us the other components of the Konishi anomaly. In this way, one may derive eqs. (6) without messing with the superfield perturbation theory. However, in these lecture notes I shall use the super-diagram approach.

Diagrammatically,

$J_{\rm ax}(V) \;=$  (11)

and the individual diagrams contributing to this amplitude look just like the $(n+1)$–vector amplitudes, except for the plus sign in eq. (1). *Naively,* these diagrams should cancel each other when we act with D^2 or $\overline{D}^2$ on the $J_{\rm ax}$ — this works similarly to the Ward identities you (should) have worked out in the homework set #4. Unfortunately, these diagrams diverge, so we must regulate them first and only then check if they really cancel each other or only seem to cancel. It turns out that all the UV regulators would spoil the cancellation of some one-loop diagrams, but the net mis-cancellation is the same for any UV regulator, hence the Konishi anomaly (6).

In these notes, I am going to derive the anomaly eqs. (6) at the one-loop level using the Pauli–Villars regulator.⋆ This means adding to the theory some very heavy fields with wrong norm in the Hilbert space and hence wrong sign for each heavy loop. In our case, we add a pair of charged chiral superfields X and Y, so

$$\begin{aligned}\mathcal{L}^{\rm reg} \;&=\; \int\! d^4\theta\left(\tfrac{1}{8}V D^\alpha \overline{D}^2 D_\alpha V \;+\; \overline{A}\, e^{+2gV} A \;+\; \overline{B}\, e^{-2gV} B\right)\\ &\qquad +\int\! d^4\theta\left(\overline{X}\, e^{+2gV} X \;+\; \overline{Y}\, e^{-2gV} Y\right) \;+\; \Lambda\!\int\! d^2\theta\, XY \;+\; \text{H.c.}\end{aligned}\tag{12}$$

Consequently, the regulated vector and axial currents become

$$J^{\rm reg}_{\rm vec} \;=\; \overline{A}\, e^{+2gV} A \;-\; \overline{B}\, e^{-2gV} B \;+\; \overline{X}\, e^{+2gV} X \;-\; \overline{Y}\, e^{-2gV} Y,\tag{13}$$

$$J^{\rm reg}_{\rm ax} \;=\; \overline{A}\, e^{+2gV} A \;+\; \overline{B}\, e^{-2gV} B \;+\; \overline{X}\, e^{+2gV} X \;+\; \overline{Y}\, e^{-2gV} Y.\tag{14}$$

The classical equations of motion for the charged fields are

$$\begin{aligned}\overline{D}^2\left(\overline{A}\, e^{+2gV}\right) \;&=\; \overline{D}^2\left(\overline{B}\, e^{-2gV}\right) \;=\; 0,\\ \overline{D}^2\left(\overline{X}\, e^{+2gV}\right) \;&=\; 4\Lambda Y, \qquad \overline{D}^2\left(\overline{Y}\, e^{-2gV}\right) \;=\; 4\Lambda X,\\ D^2\left(e^{+2gV} A\right) \;&=\; D^2\left(e^{-2gV} B\right) \;=\; 0,\\ D^2\left(e^{+2gV} X\right) \;&=\; 4\Lambda^*\overline{Y}, \qquad D^2\left(e^{-2gV} Y\right) \;=\; 4\Lambda^*\overline{X},\end{aligned}\tag{15}$$

hence *classically*

$$D^2 J^{\rm reg}_{\rm vec} \;=\; \overline{D}^2 J^{\rm reg}_{\rm vec} \;=\; 0\tag{16}$$

but

$$D^2 J^{\rm reg}_{\rm ax} \;=\; 8\Lambda^*\overline{XY} \qquad \text{and} \qquad \overline{D}^2 J^{\rm reg}_{\rm ax} \;=\; 8\Lambda XY.\tag{17}$$

For the quantum theory, eq. (16) means that the vector current is indeed conserved. At the

⋆ The dimensional reduction — like all flavors of dimensional regularization — is difficult to apply to amplitudes involving the $\epsilon^{\kappa\lambda\mu\nu}$ tensor, so it's rather inconvenient for calculating the anomalies.

same time, eq. (17) tells us that

$$\begin{aligned} \overline{D}^2 J^{\text{reg}}_{\text{ax}}(y,\theta) &= 8\Lambda \times \langle XY(y,\theta)\rangle\,, \\ D^2 J^{\text{reg}}_{\text{ax}}(\bar{y},\bar{\theta}) &= 8\Lambda^* \times \left\langle \overline{XY}(\bar{y},\bar{\theta})\right\rangle . \end{aligned} \tag{18}$$

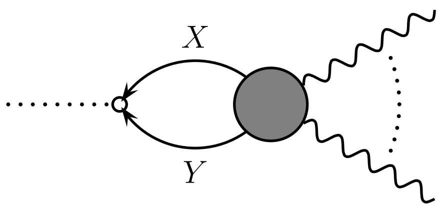
The expectation values $\langle XY\rangle$ and $\left\langle \overline{XY}\right\rangle$ vanish in the vacuum but they become not-trivial when the EM fields are present. Below, we shall see that at the one-loop level

$$\langle XY\rangle = \frac{1}{\Lambda} \times \frac{-g^2}{16\pi^2} W^\alpha W_\alpha \;+\; O(1/\Lambda^2\Lambda^*), \tag{19}$$

$$\left\langle \overline{XY}\right\rangle = \frac{1}{\Lambda^*} \times \frac{-g^2}{16\pi^2} \overline{W}_{\dot\alpha} \overline{W}^{\dot\alpha} \;+\; O(1/\Lambda^{*2}\Lambda). \tag{20}$$

Consequently, in the $\Lambda \to \infty$ limit eqs. (18) gives a finite but non-zero result for the non-conservation of the axial current, namely the Konishi anomaly (6).

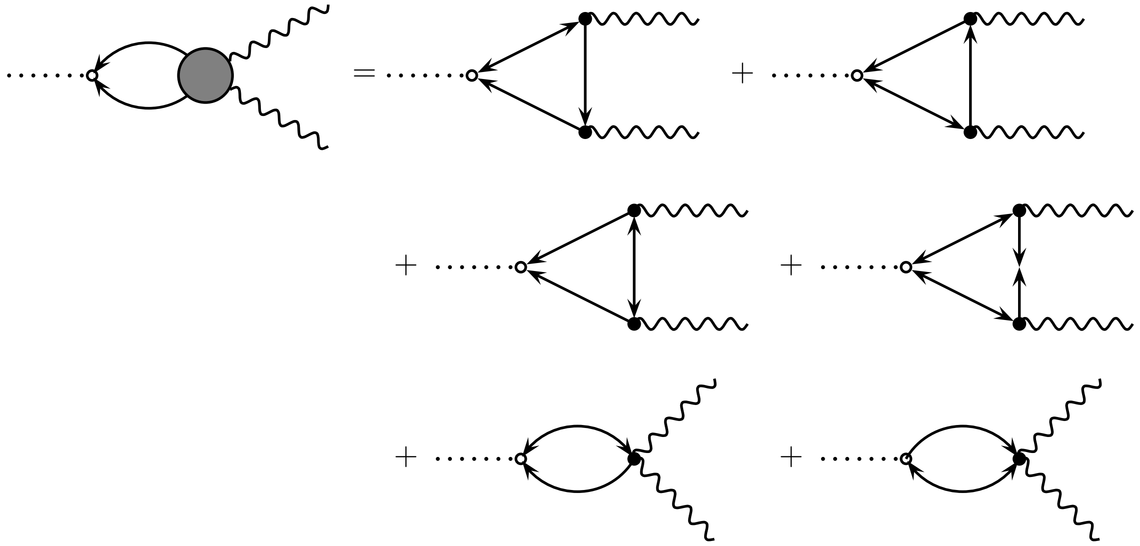
Diagrammatically, the loop corrections to the $\langle XY\rangle$ can be summarized as



(21)

Here at the ○ vertex there are no $\int d^4\theta$ or $\int d^4x$ integrals, so the amplitude has form $\langle XY\rangle =$ some composite superfield. Also, the loop (21) carries overall minus sign due to wrong norm of the Pauli–Villars fields X and Y.

By charge conjugation, the number of the external vector lines in the amplitudes (21) must be even, so let's start with the two-vector case. At the one-loop level we have 6 diagrams, namely



(22)

Note the charged fields X and Y in all these loops are very heavy — indeed their mass $|\Lambda|$ serves as the UV cutoff scale of the regulated theory. This mass is much larger than any of the external momenta p, hence

$$\frac{1}{(k+p)_E^2+|\Lambda|^2}\;\approx\;\frac{1}{k_E^2+|\Lambda|^2}\,. \tag{23}$$

for all values of the Euclidean loop momentum k ($k\sim p$, or $k\sim\Lambda$, or anything inbetween). This allows us to approximate all the propagators in the loops (22) as having the same momentum k, at least in the denominator. Thus, for the last two diagrams in eq. (22) we have

$$\begin{aligned}\langle XY\rangle_{5+6}\;=\;&-\frac{4ig^2}{2}\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2-|\Lambda|^2+i0}\right)^2\\ &\times\left(\frac{\Lambda^*\overline{D}^2}{4}\,V^2\,\frac{D^2\overline{D}^2}{16}\,\delta(\cdots)\;+\;\frac{\overline{D}^2D^2}{16}\,V^2\,\frac{\Lambda^*\overline{D}^2}{4}\,\delta(\cdots)\right)\end{aligned} \tag{24}$$

where $\delta(\cdots)$ stands for $\delta^{(4)}(\theta-\theta')$ which should be evaluates at $\theta=\theta'$ after the action of the spinor derivatives. Similarly, the first four diagrams (22) produce

$$\langle XY\rangle_{1+2+3+4} \;=\; +4ig^2\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2-|\Lambda|^2+i0}\right)^3 \times \begin{pmatrix} \frac{\Lambda^*\overline{D}^2}{4}\,V\,\frac{D^2\overline{D}^2}{16}\,V\,\frac{D^2\overline{D}^2}{16}\,\delta(\cdots) \\ +\,\frac{\overline{D}^2D^2}{16}\,V\,\frac{\overline{D}^2D^2}{16}\,V\,\frac{\Lambda^*\overline{D}^2}{4}\,\delta(\cdots) \\ -\,\frac{\overline{D}^2D^2}{16}\,V\,\frac{\Lambda^*\overline{D}^2}{4}\,V\,\frac{D^2\overline{D}^2}{16}\,\delta(\cdots) \\ -\,\frac{\Lambda^*\overline{D}^2}{4}\,V\,\frac{\Lambda D^2}{4}\,V\,\frac{\Lambda^*\overline{D}^2}{4}\,\delta(\cdots) \end{pmatrix}. \tag{25}$$

Altogether, we have

$$\langle XY\rangle \;=\; \frac{ig^2\Lambda^*}{256}\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2-|\Lambda|^2+i0}\right)^3\times\mathcal{F}\delta(\cdots) \tag{26}$$

where

$$\begin{aligned}\mathcal{F} \;=\; &\overline{D}^2VD^2\overline{D}^2VD^2\overline{D}^2 \;+\; \overline{D}^2D^2V\overline{D}^2D^2V\overline{D}^2 \;-\; \overline{D}^2D^2V\overline{D}^2VD^2\overline{D}^2\\ &-\;16|\Lambda|^2\times\overline{D}^2VD^2V\overline{D}^2 \;-\; 8(k^2-|\Lambda|^2)\times\left(\overline{D}^2V^2D^2\overline{D}^2 \;+\; \overline{D}^2D^2V^2\overline{D}^2\right).\end{aligned} \tag{27}$$

Now let's simplify this formula. The three terms on the top line here can be combined together as

$$\overline{D}^2[D^2,V]\overline{D}^2[D^2,V]\overline{D}^2 \;+\; \overline{D}^2V\times D^2\overline{D}^2D^2\times V\overline{D}^2 \tag{28}$$

where in the second term we may simplify $D^2\overline{D}^2D^2=16k^2\times D^2$. Consequently,

$$\begin{aligned}\mathcal{F} \;&=\; \overline{D}^2[D^2,V]\overline{D}^2[D^2,V]\overline{D}^2 \;+\; 16(k^2-|\Lambda|^2)\times\overline{D}^2VD^2V\overline{D}^2\\ &\qquad-\;8(k^2-|\Lambda|^2)\times\left(\overline{D}^2V^2D^2\overline{D}^2 \;+\; \overline{D}^2D^2V^2\overline{D}^2\right)\\ &=\; \overline{D}^2[D^2,V]\overline{D}^2[D^2,V]\overline{D}^2 \;-\; 8(k^2-|\Lambda|^2)\times\overline{D}^2[[D^2,V],V]\overline{D}^2.\end{aligned} \tag{29}$$

Note the commutators on the bottom line here

$$\begin{aligned}[D^2,V] \;&=\; 2(D^\alpha V)D_\alpha \;+\; (D^2V),\\ [[D^2,V],V] \;&=\; 2(D^\alpha V)(D_\alpha V)\end{aligned} \tag{30}$$

make some of the D^α operators act on the vector field V instead of the $\delta(\cdots)$ to the right of $\mathcal{F}$. Consequently, in the second term in (29)

$$\overline{D}^2[[D^2,V],V]\overline{D}^2\delta(\cdots) \;=\; 2\overline{D}^2(D^\alpha V)(D_\alpha V)\overline{D}^2\delta(\cdots) \;=\; 0 \tag{31}$$

because none of the D^α acts on the $\delta(\cdots)$. As to the first term in (29), we need two D operators to act on the $\delta(\cdots)$, hence

$$\begin{aligned}
\overline{D}^2[D^2,V]\overline{D}^2[D^2,V]\overline{D}^2\delta(\cdots) &= 4\overline{D}^2(D^\alpha V)D_\alpha\overline{D}^2(D^\beta V)D_\beta\overline{D}^2\delta(\cdots) \;+\; 0\\
&= -4\overline{D}^2(D^\alpha V)\overline{D}^2(D^\beta V)\times D_\alpha D_\beta\overline{D}^2\delta(\cdots)\\
&= -4(\overline{D}^2D^\alpha V)(\overline{D}^2D^\beta V)\times D_\alpha D_\beta\overline{D}^2\delta(\cdots)\\
&= -4(\overline{D}^2D^\alpha V)(\overline{D}^2D^\beta V)\times 8\epsilon_{\alpha\beta}\\
&= -64W^\alpha W^\beta\times 8\epsilon_{\alpha\beta}\\
&= -512W^\alpha W_\alpha\,.
\end{aligned} \tag{32}$$

To summarize,

$$\mathcal{F}\delta(\cdots) \;=\; -512W^\alpha W_\alpha\,. \tag{33}$$

The rest of the formula (26) is a straightforward integral

$$\begin{aligned}
I &= +\frac{ig^2\Lambda^*}{256}\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2-|\Lambda|^2+i0}\right)^3\\
&= +\frac{g^2\Lambda^*}{256}\int\frac{d^4k_E}{(2\pi)^4}\,\frac{1}{(k_E^2+|\Lambda|^2)^3}\\
&= +\frac{g^2\Lambda^*}{256}\times\frac{1}{16\pi^2}\int_0^\infty dk_E^2\,\frac{k_E^2}{(k_E^2+|\Lambda|^2)^3}\\
&= +\frac{g^2\Lambda^*}{256}\times\frac{1}{32\pi^2|\Lambda|^2}\\
&= \frac{g^2}{2^{13}\pi^2}\times\frac{1}{\Lambda}\,.
\end{aligned} \tag{34}$$

Combining this result with eq. (33) immediately gives us

$$\langle XY\rangle\,(2\text{ vectors}) \;=\; \frac{-g^2}{16\pi^2\Lambda}\,W^\alpha W_\alpha\,. \tag{35}$$

To complete the proof of eq. (19) we need to show that there are no multi-vector contributions to the $\langle XY\rangle$. Or rather, that all amplitudes (21) involving $n=4,6,\ldots$ vectors are smaller than $O(1/\Lambda)$. Although the number of diagrams increases rather rapidly with n — for example, for $n = 4$ there are 54 one-loop diagrams — one can prove by induction that all the vector fields appear in the analogue of $\mathcal{F}$ only in commutators $[D^2, V]$ or multiple commutators. Consequently, although there are up to $2n$ D^α operators in the loop, at least n of them act on the vector fields while two more have to act on the $\delta(\cdots)$. This leaves us no more then $n-2$ D's to anticommute with the $\overline{D}$'s and produce powers of the loop momentum in the numerator. Altogether, the loop integral looks like

$$I_n \;=\; \int \frac{d^4k_E}{(2\pi)^4}\,\frac{\Lambda^*\mathcal{N}_n(k_E^2,|\Lambda|^2)}{(k_E^2+|\Lambda|^2)^{n+1}} \tag{36}$$

where $\mathcal{N}_n$ in the numerator is some polynomial of degree $(n-2)/2$. By the power-of-momentum counting,

$$I_n \;\sim\; \frac{\Lambda^*}{|\Lambda|^n} \qquad \Longrightarrow\ \forall n>2,\ \Lambda\times I_n \to 0 \text{ when } \Lambda\to\infty. \tag{37}$$

In other words, all the multi-vector terms eq. (19) are sub-leading in the $\Lambda\to\infty$ limit and only the two-vector term contributes to the Konishi anomaly

$$\overline{D}^2 J_{\rm ax}^{\rm reg} \;=\; 8\Lambda\,\langle XY\rangle \;\xrightarrow[\Lambda\to\infty]{}\; \frac{-g^2}{2\pi^2}\,W^\alpha W_\alpha\,. \tag{38}$$

Similar arguments show that

$$\left\langle\overline{XY}\right\rangle(2\text{ vectors}) \;=\; \frac{-g^2}{16\pi^2\Lambda}\,\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}. \tag{39}$$

while the multi-vector contributions to eq. (20) carry sub-leading powers of $1/|\Lambda|$, hence

$$D^2 J_{\rm ax}^{\rm reg} \;=\; 8\Lambda^*\left\langle\overline{XY}\right\rangle \;\xrightarrow[\Lambda\to\infty]{}\; \frac{-g^2}{2\pi^2}\,\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}. \tag{40}$$

The details are left as an exercise to the students.