1. Consider supersymmetric QCD with N_c colors and N_f flavors. In matrix notations, the quark chiral superfields $A_i^{\,f}(y,\theta)$ form an $N_c \times N_f$ matrix A while the antiquark chiral superfields $B_f^{\,i}(y,\theta)$ form an $N_f \times N_c$ matrix B. Let all the flavors be exactly massless, so the Lagrangian is

$$\mathcal{L} \;=\; \frac{i\tau}{8\pi}\int d^2\theta\,\operatorname{tr}(W^\alpha W_\alpha) \;+\; \mathrm{H.c.} \;+\int d^4\theta\,\operatorname{tr}\left(\overline{A}\,e^{+2V}A \,+\, B\,e^{-2V}\overline{B}\right). \tag{1}$$

(a) Show that

$$V_{\text{scalar}} \;=\; \frac{g^2}{8}\sum_{a=1}^{N^2-1}\left[\operatorname{tr}\left(\lambda^a\big(AA^\dagger - B^\dagger B\big)\right)\right]^2. \tag{2}$$

(b) Show that this potential vanishes if and only if

$$AA^\dagger \;-\; B^\dagger B \;=\; c\times \mathbf{1}_{\mathrm{N_c}\times\mathrm{N_c}} \tag{3}$$

for some real number c. Also show that for $N_f < N_c$ this matrix relation implies $c=0$ and hence

$$AA^\dagger \;=\; B^\dagger B. \tag{4}$$

(c) Show that all solutions to eqs. (4) have form

$$A \;=\; U_C\times\begin{pmatrix}\mathbf{D}_{N_f\times N_f}\\ \hline \mathbf{0}_{(N_c-N_f)\times N_f}\end{pmatrix}\times V_A\,, \qquad B \;=\; V_B\times\left(\mathbf{D}_{N_f\times N_f}\,\Big|\,\mathbf{0}_{N_F\times(N_c-N_f)}\right)\times U_C^{-1} \tag{5}$$

where U_C is an $SU(N_c)$ matrix (same gauge symmetry for A and B), V_A and V_B are $N_F\times N_F$ unitary matrices, and $\mathbf{D}$ is a real ≥ 0 diagonal $N_f\times N_F$ matrix, same $\mathbf{D}=\operatorname{diag}(d_1,\ldots d_{N_f})$ for both A and B.

(d) The independent holomorphic moduli of the flat directions form an $N_f \times N_f$ matrix $\mathcal{M} = BA$. (We assume $N_f < N_c$.) Use eqs. (5) to argue that this moduli matrix indeed uniquely determined a point in the moduli space, *i.e.*, a pair of matrices A and B matrices satisfying eq. (4) *modulo an $SU(N_c)$ gauge symmetry.* In other words, pairs (A, B) and $(UA, BU^\dagger)$ related by a gauge symmetry U count as the same point in the moduli space.

2. Now consider the superfield Feynman rules for the Wess–Zumino model or a more general theory that has only chiral superfields (and their antichiral conjugates) and all the interactions come from the superpotential. Let's count the fermionic derivative operators D^α and $\overline{D}^{\dot\alpha}$ in a generic Feynman diagram, tree or loop. This counting should be done *before* you do the Grassmannian integrals and use up some derivatives to close the loops via $D^2\overline{D}^2\delta^{(4)}\left(\theta_1 - \theta_2\right)|_{\theta_1=\theta_2} = 16$, *etc.*.

   Show that the net number of the fermionic derivatives is

$$\#(D^\alpha) \;+\; \#(\overline{D}^{\dot\alpha}) \;=\; 2\#(\text{loops}) \;+\; 2\#(\Phi\overline{\Phi}\text{ propagators}) \;-\; 2. \tag{6}$$

   Note that this number is non-negative for all loop graphs and also for all tree graphs that have a propagator of the $\Phi\overline{\Phi}$ type. Consequently, all such graphs yields amplitudes of the $\int d^4\theta$ form.

   The only exceptions are the tree graphs where all propagators are of the types $\Phi\Phi$ or $\overline{\Phi\Phi}$. For such graphs there is one un-cancelled $1/D^2$ or $1/\overline{D}^2$ factor from the vertices and the resulting amplitudes have form

$$\int d^4\theta\,\frac{-4}{\overline{D}^2}\Phi\times\Phi\cdots\Phi \;=\; \int d^2\theta\,\Phi\times\Phi\cdots\Phi \quad\text{or}\quad \int d^4\theta\,\frac{-4}{D^2}\overline{\Phi}\times\overline{\Phi}\cdots\overline{\Phi} \;=\; \int d^2\bar\theta\,\overline{\Phi}\times\overline{\Phi}\cdots\overline{\Phi}. \tag{7}$$

   This is how integrating out massive fields can yields superpotential terms, but only at the tree level.

3. Finally, consider supersymmetric QED,

$$\mathcal{L} = \int d^4\theta \left(\overline{A}e^{+2eV}A + \overline{B}e^{-2eV}B + \tfrac{1}{8}VD^\alpha\overline{D}^2D_\alpha V\right) + \int d^2\theta\, mAB \int d^2\bar{\theta}\, m^*\overline{AB}. \quad (8)$$

Superfield Feynman rules for SQED will be explained in class next week. For now, please take them for granted:

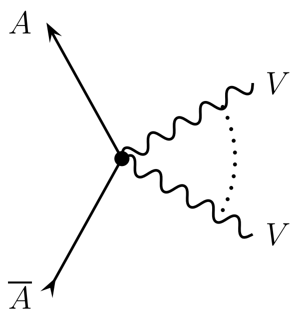
- Chiral propagators:

$$\begin{aligned}
\overline{A} \succ\!\!\longrightarrow A &= \frac{i}{p^2 - mm^* + i0} \times \frac{\overline{D}^2D^2}{16}\,\delta^{(4)}(\theta_1 - \theta_2),\\
\overline{A} \succ\!\!\longrightarrow\!\!\prec \overline{B} &= \frac{i}{p^2 - mm^* + i0} \times \frac{m\overline{D}^2}{4}\,\delta^{(4)}(\theta_1 - \theta_2),\\
B \longleftarrow\!\!\longrightarrow A &= \frac{i}{p^2 - mm^* + i0} \times \frac{m^*D^2}{4}\,\delta^{(4)}(\theta_1 - \theta_2),\\
B \longleftarrow\!\!\prec \overline{B} &= \frac{i}{p^2 - mm^* + i0} \times \frac{D^2\overline{D}^2}{16}\,\delta^{(4)}(\theta_1 - \theta_2),
\end{aligned} \quad (9)$$

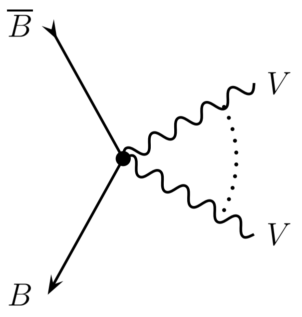
- Vector propagator in the Feynman gauge:

$$V \sim\!\sim\!\sim V = \frac{i}{k^2 + i0} \times \delta^{(4)}(\theta_1 - \theta_2). \quad (10)$$

- Vertices: One incoming chiral line, one outgoing chiral line of the same species, any number $n = 1, 2, 3, \ldots$ of vector lines,



$= i(+2e)^n$,



$= i(-2e)^n$, (11)

without any superderivative factors in the numerator or denominator.

Count the superderivatives and powers of momenta in a general Feynman diagram and show that a diagram with E_C external legs of chiral superfields (A, B, $\overline{A}$, or $\overline{B}$), E_V external legs of vectors, and any numbers of loops, vertices, and internal lines has superficial degree of divergence

$$\Delta \;\leq\; 2 \;-\; E_C\,. \tag{12}$$

In class, I shall use this formula to prove that SQED is renormalizable.