1. The first two problems are about SQCD with $N_c = N$ colors and massless $N_f = N+1$ flavors. Let's start with the classical moduli space of this theory.

   (a) Show that the classical moduli space has N_f^2 complex dimensions.

   The holomorphic gauge invariants of the quark Q_f^c and antiquark $\widetilde{Q}_{f,c}$ chiral superfields include

$$\begin{aligned} \text{mesons } \mathcal{M}_{ff'} &= \widetilde{Q}_{f,c} Q^c_{f'}\,, \\ \text{baryons } \mathcal{B}^f &= \frac{1}{N!}\epsilon^{ff_1\cdots f_N}\epsilon_{c_1\cdots c_N} Q^{c_1}_{f_1}\cdots Q^{c_N}_{f_N}\,, \\ \text{and antibaryons } \widetilde{\mathcal{B}}^f &= \frac{1}{N!}\epsilon^{ff_1\cdots f_N}\epsilon^{c_1\cdots c_N}\widetilde{Q}_{f_1,c_1}\cdots\widetilde{Q}_{f_N.c_N}\,. \end{aligned} \tag{1}$$

   But these invariants are not independent:

   (b) Show that *classically*, the invariants (1) satisfy several constraints, namely

$$\det(\mathcal{M}) \;=\; 0, \quad \mathcal{M}_{ff'}\mathcal{B}^{f'} \;=\; 0, \quad \widetilde{\mathcal{B}}^f\mathcal{M}_{ff'} \;=\; 0, \quad \operatorname{minor}(\mathcal{M})^{ff'} \;=\; \widetilde{\mathcal{B}}^f\mathcal{B}^{f'}. \tag{2}$$

   (c) Show that the space of mesons, baryons, and antibaryons which satisfy these constraints has precisely N_f^2 dimensions. Consequently, it may be identified with the classical moduli space of the SQCD with $N_f = N_c + 1$.

   In the low-energy effective theory for the moduli superfields we may treat the moduli $\mathcal{M}_{ff'}$, $\mathcal{B}^f$, $\widetilde{\mathcal{B}}^f$ as independent superfields,

$$\mathcal{L}_{\text{eff}} \;=\; \int d^4\theta\, K(\mathcal{M},\mathcal{B},\widetilde{\mathcal{B}};\overline{\mathcal{M}},\overline{\mathcal{B}},\overline{\widetilde{\mathcal{B}}}) \;+\; \int d^2\theta\, W(\mathcal{M},\mathcal{B},\widetilde{\mathcal{B}}) \;+\; \text{H.c.}, \tag{3}$$

   but the VEVs $\langle M\rangle$, $\langle B\rangle$, $\langle\widetilde{\mathcal{B}}\rangle$ must satisfy constraints $\partial W/\partial(\text{any modulus}) = 0$.

   (d) Show that constraints on the mesonic and baryonic VEVs due to effective superpotential

$$W_{\text{tree}} \;=\; C\left(\widetilde{\mathcal{B}}^f\mathcal{M}_{ff'}\mathcal{B}^f \;-\; \det(\mathcal{M})\right), \qquad C = \text{const} \tag{4}$$

   are precisely the classical constraints (2).

   Note: C has dimension $1-2N_c$, so we expect $C = \Lambda^{1-2N_c} \times$ a numerical constant.

- ⋆ In the chiral ring language, $\mathcal{M}_{ff'}$, $\mathcal{B}^f$, and $\widetilde{\mathcal{B}}^f$ are generators of the SQCD's off-shell chiral ring and eqs. (2) are operatorial identities for those generators. In the low-energy effective field theory, there are no operatorial identities; instead, eqs. (2) are the on-shell chiral ring equations which follow from the superpotential (4). Thus, the off-shell chiral ring of SQCD becomes the on-shell chiral ring of the effective theory.

To study the quantum corrections to the superpotential (4) — and hence to the complex structure of the moduli space — consider the flavor symmetries of the SQCD,

$$G_F \;=\; SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_A \times U(1)_R \tag{5}$$

(e) Describe how all these symmetries act on the moduli fields $\mathcal{M}_{ff'}$, $\mathcal{B}^f$, and $\widetilde{\mathcal{B}}^f$ and on the $\Lambda^{3N_c-N_f}$.

Note: The $U(1)_A$ and the $U(1)_R$ symmetries are anomalous, but the appropriate adjustment of the Θ angle — and hence on the $\Lambda^{3N_c-N_f} \propto \exp(-8\pi^2 f_w)$ — would cancel the anomaly.

The exact superpotential $W(\mathcal{M},\mathcal{B},\widetilde{\mathcal{B}};\Lambda^{3N_c-N_f})$ for the moduli fields of the quantum theory must be invariant under all the flavor symmetries (5).

(f) Show that this implies

$$W(\mathcal{M},\mathcal{B},\widetilde{\mathcal{B}};\Lambda^{3N_c-N_f}) \;=\; \Lambda^{1-2N_c} \times F\left(\left(\widetilde{\mathcal{B}}^f\mathcal{M}_{ff'}\mathcal{B}^{f'}\right),\det(M)\right) \tag{6}$$

where $F(x,y)$ is a holomorphic homogeneous function of degree 1, *i.e.*, $F(\alpha x,\alpha y) = \alpha F(x,y)$.

Note that the classical effective superpotential (4) is indeed of the form (6) for $F(x,y) = x - y$, provided we identify the overall coefficient C as Λ^{1-2N_C}.

In general, the quantum corrections due to instantons or other non-perturbative effects should carry higher powers of the $\Lambda^{3N_c-N_f}$ than the classical superpotential. But for the superpotential (6), the power of Λ is completely fixed by the R–symmetry, which means that there are no non-perturbative corrections at all! Instead

$$W(\mathcal{M},\mathcal{B},\widetilde{\mathcal{B}};\Lambda^{3N_c-N_f}) \;=\; W_{\text{tree}} \;+\; 0 \;=\; \Lambda^{1-2N_c} \times \left((\widetilde{\mathcal{B}}^f\mathcal{M}_{ff'}\mathcal{B}^{f'} \;-\; \det(\mathcal{M})\right), \tag{7}$$

and there are no quantum corrections to the classical constraints (2).

2. The classical moduli space of SQCD with $N_f = N_c + 1$ has a singular point $\langle Q\rangle = \langle\widetilde{Q}\rangle = 0$ where none of the symmetries are broken. In problem 1 we saw that the quantum moduli space of the theory has the same complex structure, so it has a similar singular point $\mathcal{M} = \mathcal{B} = \widetilde{\mathcal{B}} = 0$ where all the flavor symmetries remain unbroken despite the color confinement. Or rather, all the flavor symmetries free from the color (CCF) anomalies remain unbroken.

   (a) Show that a combination of the axial symmetry $U(1)_A$ and the R–symmetry $U(1)_R$ which acts on the quarks, antiquarks, gluinos, and their superpartners as

$$\begin{aligned} \lambda^\alpha &\to e^{i\rho}\lambda^\alpha, \quad \Psi^\alpha \to e^{-i(N_c/N_f)\rho}\Psi^\alpha, \quad \widetilde{\Psi}^\alpha \to e^{-i(N_c/N_f)\rho}\widetilde{\Psi}^\alpha, \\ A^\mu &\to A^\mu, \quad Q \to e^{i\rho/N_f}Q, \quad \widetilde{Q} \to e^{i\rho/N_f}\widetilde{Q} \end{aligned} \tag{8}$$

   is free from the color anomaly. Consequently, the net color-anomaly-free flavor symmetry is

$$\hat{G}_f = SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_{RA} \tag{9}$$

   where $U(1)_B$ is the vector-like baryon number and $U(1)_{RA}$ is the symmetry (8).

At the singular point of the moduli space, the entire flavor symmetry (9) of SQCD remains unbroken, which calls for 't Hoof's anomaly matching condition between the elementary and composite fermions. The elementary fermions here are the quarks, the antiquarks, and the gluinos, while the massless composite fermions are the fermionic superpartners of the massless moduli $\mathcal{M}_{ff'}$, $\mathcal{B}^f$, and $\widetilde{\mathcal{B}}^f$.

   (b) In the effective theory of the $N_f^2 + 2N_f$ chiral superfields $\mathcal{M}_{ff'}$, $\mathcal{B}^f$, $\widetilde{\mathcal{B}}^f$, at a generic point of the moduli space, the superpotential (7) makes $2N_f$ superfields massive while the remaining N_f^2 remain massless. But at the singular point $\mathcal{M} = \mathcal{B} = \widetilde{\mathcal{B}} = 0$, all the $N_f^2 + 2N_f$ superfields remain massless and their fermionic components contribute to the flavor anomalies. Prove this.

   (c) List the flavor (9) quantum numbers of all the massless composite fermions. For comparison, list the flavor and color quantum numbers of the elementary fermions.

   (d) And now comes the hard part: Calculate all the non-trivial flavor anomalies $\mathrm{tr}(F)$ and $\mathrm{tr}(F\{F', F''\})$ over the elementary fermions and over the massless composite fermions and verify that in all cases

$$\mathrm{tr}_{\text{elem}}(F) \;=\; \mathrm{tr}_{\text{comp}}(F), \quad \mathrm{tr}_{\text{elem}}(F\{F',F''\}) \;=\; \mathrm{tr}_{\text{comp}}(F\{F',F''\}) \quad \forall F,F',F'' \in \hat{G}_f\,. \tag{10}$$

3. The last problem is about the $U(N)$ gauge theory with a single adjoint multiplet Φ of chiral matter. We are interested in the off-shell chiral ring of the theory, which is made out of gauge-invariant combinations of the scalar and gaugino fields — or in superfield notations, out of Φ and W^α chiral superfields.

   In matrix notations, all such gauge invariant combinations are polynomials in the traces of matrix products of matrix products of the Φ and W_α matrices,

$$T_k \;=\; \mathrm{tr}\bigl(\Phi^k\bigr), \quad P_k^\alpha \;=\; \mathrm{tr}\bigl(\Phi^k W^\alpha\bigr), \quad R_{k,\ell}^{\alpha\beta} \;=\; \mathrm{tr}\bigl(\Phi^k W^\alpha \Phi^\ell W^\beta\bigr), \quad \textit{etc.}, \tag{11}$$

   or in other words, the traces (11) *generate* the chiral ring of the theory.

   Actually, many of the traces (11) are equivalent to each other as members of the chiral ring because their difference is a total $\overline{D}^{\dot\alpha}$ derivative of some gauge-invariant operator. Your task is to show that the only independent generators are the T_k, P_k^α, and $R_k = R_{k,0}^{12}$, and there are no others.

   (a) Show that for any matrix product X of Φ and W^α matrices,

$$\mathrm{tr}\bigl(X[W^\alpha,\Phi]\bigr) \;=\; -\tfrac{1}{8}\overline{D}^2\,\mathrm{tr}\bigl(X\nabla^\alpha\Phi\bigr) \tag{12}$$

   where $\nabla^\alpha\Phi = D^\alpha\Phi + [\Gamma^\alpha,\Phi]$ (where $\Gamma^\alpha = e^{-2V}D^\alpha e^{2V}$) is the gauge-covariant spinor derivative, and hence

$$\mathrm{tr}\bigl(XW^\alpha\Phi\bigr) \;\stackrel{\text{c.r.}}{=}\; \mathrm{tr}\bigl(X\Phi W^\alpha\bigr). \tag{13}$$

   Note that (a) allows us to re-order the *Phi* and W^α matrices in the traces (11), thus

$$\mathrm{tr}\bigl(\Phi\cdots\Phi W^\alpha\Phi\cdots\Phi W^\beta\cdots\text{more matrices}\cdots W^\gamma\bigr) \;\stackrel{\text{c.r.}}{=}\; \mathrm{tr}\bigl(\Phi\cdots\Phi\times W^\alpha W^\beta\cdots W^\gamma\bigr). \tag{14}$$

   In particular, $R_{k,\ell}^{\alpha\beta} \stackrel{\text{c.r.}}{=} R_{k+\ell,0}^{\alpha\beta}$.

(b) Now show that

$$\mathrm{tr}\big(X\{W^\alpha, W^\beta\}\big) \;=\; -\tfrac{1}{8}\overline{D}^2\,\mathrm{tr}\big(X\nabla^\alpha W^\beta\big) \tag{15}$$

and hence

$$\mathrm{tr}\big(\Phi^k W^\alpha W^\beta\big) \;\overset{\text{c.r.}}{=}\; \epsilon^{\alpha\beta}\,\mathrm{tr}\big(\Phi^k W^1 W^2\big) \;=\; \tfrac{1}{2}\epsilon^{\alpha\beta}\times R_k\,. \tag{16}$$

(c) Finally, show that all traces including 3 or more gaugino fields W^α are equivalent to zero,

$$\mathrm{tr}\big(\Phi^k W^\alpha W^\beta W^\gamma\cdots\big) \;\overset{\text{c.r.}}{=}\; 0. \tag{17}$$

Thus, the only independent generators of the chiral ring are the

$$T_k \;=\; \mathrm{tr}\big(\Phi^k\big),\quad P_k^\alpha \;=\; \mathrm{tr}\big(\Phi^k W^\alpha\big),\quad \text{and}\quad R_k \;=\; \mathrm{tr}\big(\Phi^k W^\alpha W_\alpha\big). \tag{18}$$

Moreover, for finite N, the traces involving more then N Φ matrices are polynomial functions of the traces with fewer Φ's. For example, for $N=2$

$$\begin{aligned} 2T_3 &= 3T_2T_1 - T_1^3, \\ 2P_3^\alpha &= P_2^\alpha\times T_1 + P_1^\alpha\times T_2 - P_0^\alpha\times(T_1^2-T_2)T_1\,, \\ 2R_3 &= R_2\times T_1 + R_1\times T_2 - R_0\times(T_1^1-T_2)T_1\,, \end{aligned} \tag{19}$$

and similar (but more complicated) relations for $k=4,5,\ldots$. In general, such relations are corrected by the instanton effects, but that goes beyond the scope of this exercise.