1. First, a reading assignment: Gerard 't Hooft's lecture notes on *Monopoles, Instantons, and Confinement,* arXiv:hep-th/0010225. These are the same notes I have assigned back in the homework#8, but this time please focus on chapter 3 about the magnetic monopoles and §4.8 about the Θ angle in QED.

2. And now an exercise in Seiberg duality. As discussed in class, theory A is pure SQCD while theory B is SQCD with N_f^2 extra singlets $\Phi^{ff'}$, the flavor number N_f is the same in both theories, and the respective color numbers are related as $N_c^B = N_f - N_c^A$. In class, I have identified the mesons $\mathcal{M}^{ff'}$ of the A theory with the elementary color-singlet fields $\Phi^{ff'}$ of the B theory, but the literal identification $\mathcal{M}^{ff'} = \Phi^{ff'}$ works only for the renormalized fields of the same non-canonical dimension $\Delta = 3N_c^B/N_f$. In terms of the un-renormalized fields which keep their canonical dimensions — 1 for the elementary $\Phi^{ff'}$ but 2 for the mesons $\mathcal{M}^{ff'} = \widetilde{Q}_c^f Q^{cf'}$ — the duality has form

$$\mathcal{M}^{ff'} \;=\; \Phi^{ff'} \times \mu \tag{1}$$

for some constant μ of dimension +1. In this exercise, you will see that the value of this constant should be

$$\mu \;=\; \lambda \times \left((-1)^{N_c^A} \Lambda_A^{3N_c^A - N_f} \times \Lambda_B^{3N_c^B - N_f}\right)^{1/N_f} \tag{2}$$

where λ is the un-normalized Yukawa coupling of $\Phi^{ff'}$ to $\tilde{q}_{fc}$ and q_f^c in the B theory. Note that for $N_c^A + N_c^B = N_f$ this μ indeed has dimension 1.

A comment on Λ_A and Λ_B is in order here. Inside the conformal window $\frac{1}{3}N_f < N_c^A, N_c^B < \frac{2}{3}N_f$, both A and B theories are asymptotically free: their gauge couplings are weak at very high energies but become strong at the respective *infrared* scales Λ_A and Λ_B. Outside the conformal window, one of the two theories has $\beta_g > 0$ for weak g, so it's weakly coupled in the IR but becomes strongly coupled in the UV; for this theory, Λ denotes the *ultraviolet* scale at which the coupling becomes strong. Eq. (2) applies both inside and outside the conformal window.

Suppose the color-to-flavor ratio of the A theory is in the conformal window but all the quark flavors are massive,

$$W^A_{\text{tree}} \;=\; m_{ff'}\widetilde{Q}^f_c Q^{cf'}, \qquad \text{rank}(m) = N_f\,. \tag{3}$$

The masses spoil the conformal invariance of the IR physics; instead, we have a mass gap (no massless particles at all) due to quark confinement and gaugino condensation.

(a) Calculate the gaugino condensate $\langle S\rangle$ and the meson VEVs $\left\langle \mathcal{M}^{ff'}\right\rangle$ in the massive theory A. Note: when all quark flavors are massive, quantum corrections lead to $\text{rank}(\mathcal{M}) = N_f$ contrary to the classical limit $\text{rank}(\mathcal{M}_{\text{cl}}) \leq N_c$.

Under Seiberg duality, the quark masses (3) of the A theory are dual to the O'Raifeartaigh terms for the Φ fields of the B theory. In light of eq. (1), the un-normalized superpotential of the B theory is

$$W^B_{\text{tree}} \;=\; \mu m_{ff'}\Phi^{ff'} \;+\; \lambda\Phi^{ff'}\tilde{q}_{fc}\, q^c_{f'}\,. \tag{4}$$

(b) Find the supersymmetric vacua of the B theory and calculate the VEVs of the gaugino condensate S_B, the B-mesons $M^B_{ff'} = \tilde{q}_{fc}\, q^c_{f'}$, and the $\Phi^{ff'}$ fields. Hint: first derive the chiral ring equations

$$\begin{aligned} \lambda M^B_{ff'} &= -\mu m_{ff'}\,, \\ \lambda\Phi^{ff'} M^B_{ff''} &= -\delta^f_{f''} \times S_B\,, \\ \left(S^{N_c}\right)_B &= \left(\Lambda^{3N_c-N_f}\right)_B \times \det(-\lambda\Phi), \end{aligned} \tag{5}$$

then solve them.

(c) Show that if we take μ as in eq. (2) then $\langle S\rangle_B = -\langle S\rangle_A$ and $\mu\left\langle \Phi^{ff'}\right\rangle_B = \left\langle M^{ff'}\right\rangle_A$.

Now suppose $N^A_c \leq \frac{1}{3}N_f$ so the A theory is not asymptotically free but has a Landau pole at some UV scale Λ_A. The gauge coupling is strong at $E \sim \Lambda_A$, becomes weaker at lower energies until we reach the quark mass scale m, but then the β_g changes sign and the coupling becomes strong again at some very low energies. Eventually, we have gaugino condensation and confinement, similarly to the asymptotically free theories with $N_c > \frac{1}{3}N_f$.

(d) Calculate the gaugino condensate and the meson VEVs for $N_c^A \leq \frac{1}{3}N_f$ and compare them to the similar VEVs in the Seiberg-dual theory B. Show that for μ as in eq. (2) you get $\langle S\rangle_A = -\langle S\rangle_B$ and $\left\langle M^{ff'}\right\rangle_A = \mu\left\langle \Phi^{ff'}\right\rangle_B$.

Finally, suppose $N_f > N_c^A \geq \frac{2}{3}N_f$ while $N_c^B \leq \frac{1}{3}N_f$. Since the B theory is both not-AF and completely Higgsed down, it is hard to see how it can generate a gaugino condensate of a meson VEV matrix of rank N_f. On the other hand, we expect the non-perturbative effects at the UV scale Λ_B to generate some non-renormalizable $O(\Lambda_B^{\text{negative}})$ superpotential terms for the Φ fields in addition to (4),

$$W^B \;=\; \mu m_{ff'}\Phi^{ff'} \;+\; \lambda\Phi^{ff'}\tilde{q}_{fc}\,q^c_{f'} \;+\; W_{\text{np}}(\Phi,\Lambda_B). \tag{6}$$

(e) For $\mu m \ll \Lambda_B^2$, the non-perturbative term should have all the flavor symmetries of the massless B theory (including the anomalous symmetries which affect the $\Lambda_B^{3N_c^B-N_f}$). Show that these symmetries constrain the non-perturbative term to be

$$W_{\text{np}} \;=\; C'\left(\frac{\det(\lambda\Phi)}{\Lambda_B^{N_f-3N_c^B}}\right)^{1/N_c^B} \;=\; -N_c^B\left(\frac{\det(-\lambda\Phi)}{C\Lambda_B^{N_f-3N_c^B}}\right)^{1/N_c^B}, \tag{7}$$

for some numeric $O(1)$ constants C' and C

(f) Show that the B theory with superpotential (6) has $N_c^A = N_f - N_c^B$ SUSY vacua and that the VEVs $\left\langle \mu\Phi^{ff'}\right\rangle$ for those vacua agree with the meson VEVs $\left\langle \mathcal{M}^{ff'}\right\rangle$ of the A theory.

Note: for the purposes of this part of the problem, absorb the C factor in eq. (7) into a redefinition of the Λ_B, $C\Lambda_B^{N_f-3N_c} \to \Lambda_B^{N_f-3N_c}$.