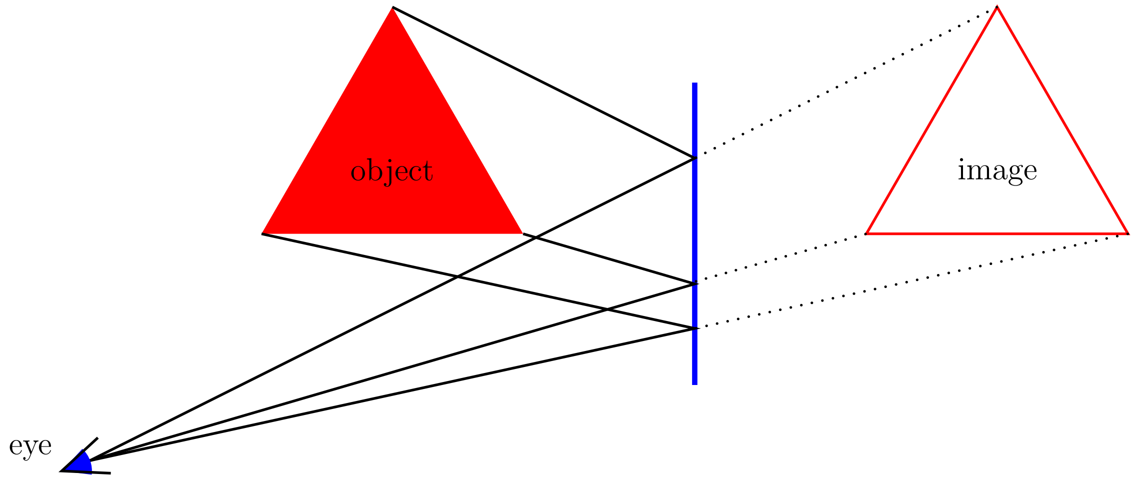
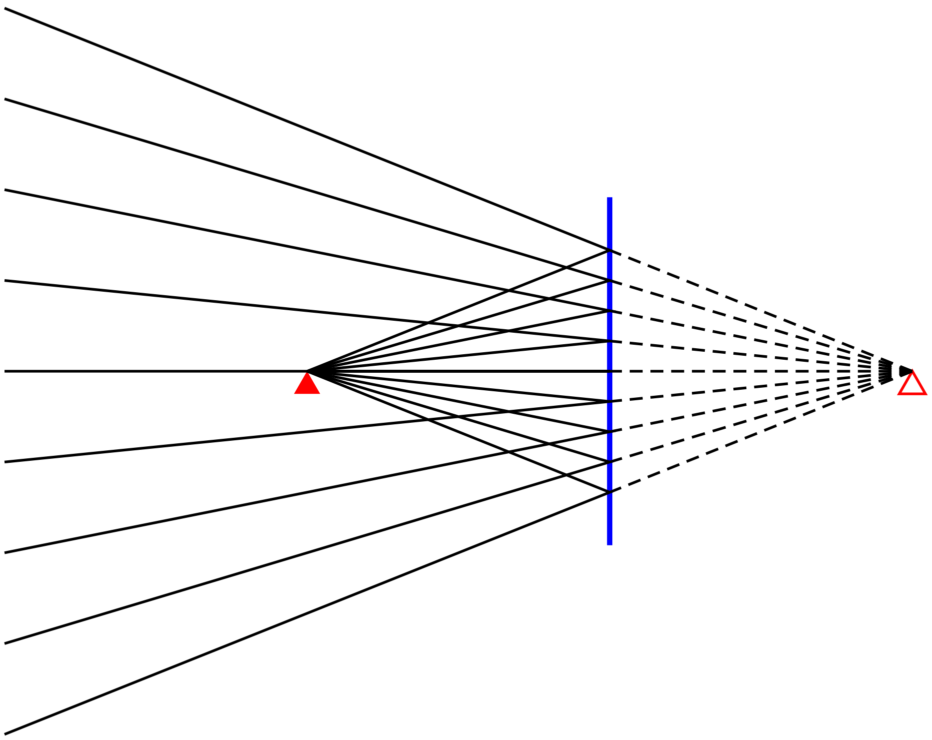
# Mirrors and Images

## Plane mirrors

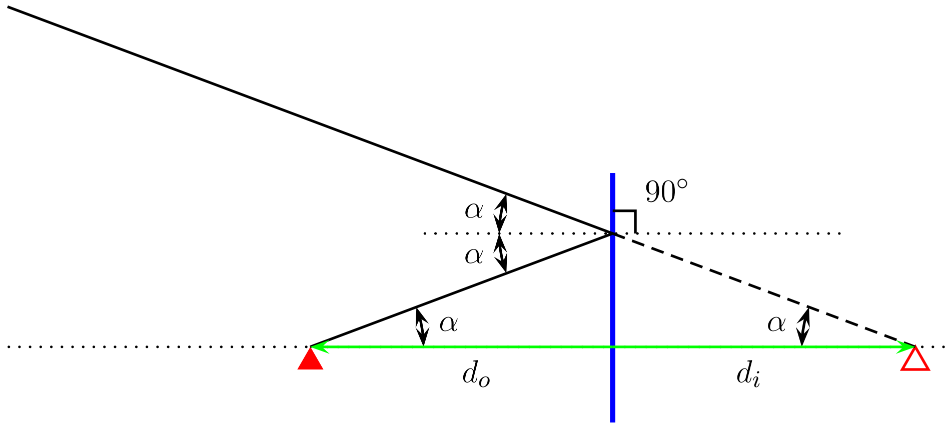
Image in a mirror:




Same image from many points of view:



Ray geometry and distances from the mirror:




Since the angle of reflection is always equal to the angle of incidence, all the angles here denoted by $\alpha$ are equal to each other. Consequently, the distance $d_i$ between the image and the mirror is equal to the distance $d_o$ between the mirror and the object. Indeed, simple trigonometry for the point $(0, y)$ where the ray is reflected off the mirror gives us two equations:

$$y \;=\; d_0 \times \tan\alpha \;=\; d_i \times \tan\alpha \tag{1}$$

and since they both hold for the same angle $\alpha$, we must have

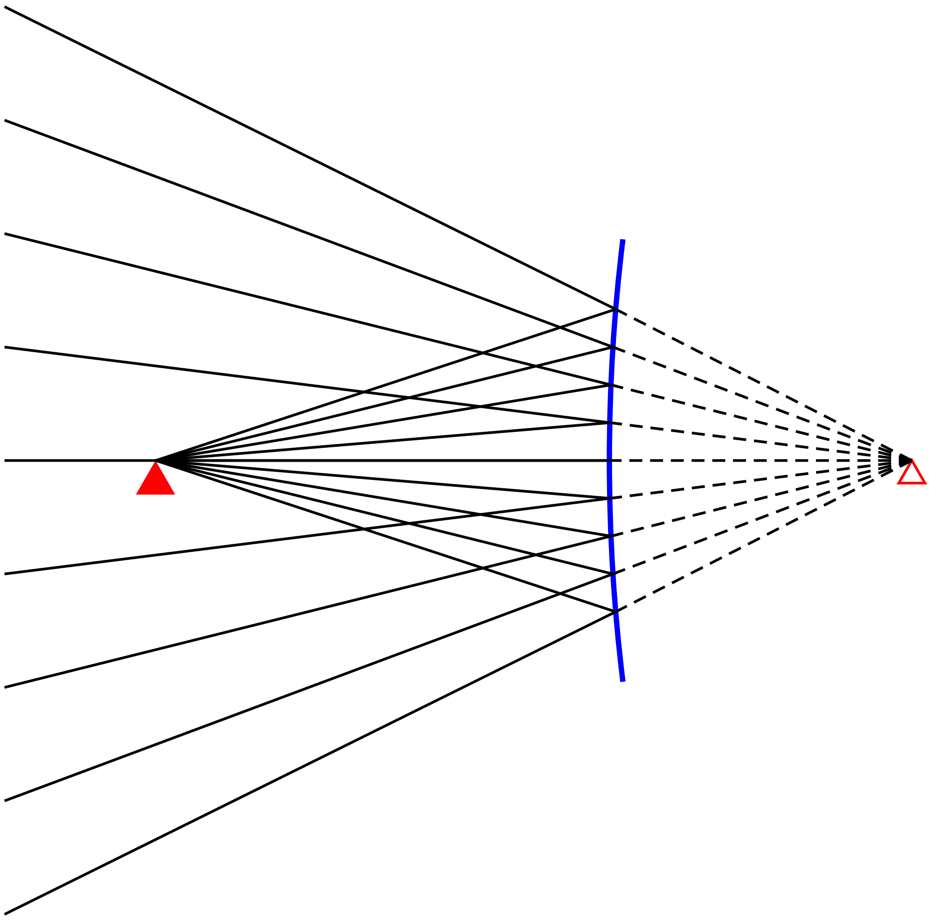
$$d_i \;=\; d_0 \quad \text{for a plane mirror.} \tag{2}$$

However, this simple formula does not work for curved mirrors. Instead, the image in a convex mirror is closer than the object, while the image in a concave mirror is more distant than the object:

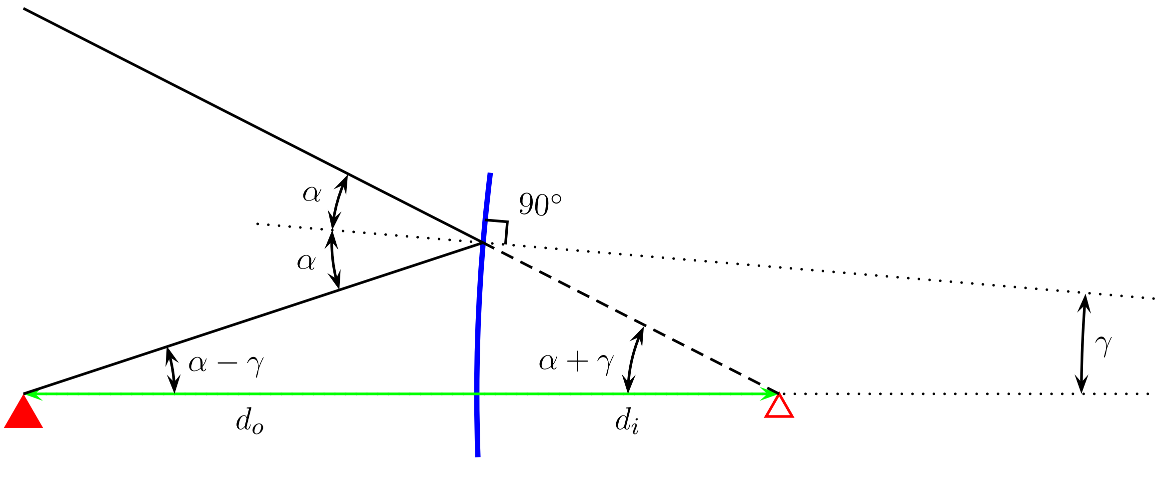
$$d_i \;<\; d_o \quad \text{for a convex mirror,}$$
$$d_i \;>\; d_o \quad \text{for a concave mirror.}$$

## Convex Mirror

Here is the ray diagram for a convex mirror:



And here is the detailed geometry for one particular ray



On this diagram, $\alpha$ is the angle of incidence = angle of reflection and $\gamma$ is the angle between the mirror's surface and the $y$ axis. Because of $\gamma \neq 0$ for a curved mirror, the angle between the incident ray and the $x$ axis is $\alpha - \gamma$ instead of $\alpha$, while the angle between the $x$ axis and the reflected ray is $\alpha + \gamma$. Consequently, for this ray geometry

$$\begin{aligned} \tan(\alpha - \gamma) &= \frac{y}{d_o + x}, \\ \tan(\alpha + \gamma) &= \frac{y}{d_i - x}, \\ \sin\gamma &= \frac{y}{R}, \end{aligned} \tag{3}$$

where $(x, y)$ are the coordinates of the point where the ray is reflected off the mirror and $R$ is the mirror's radius of curvature. For small $y$ — and hence $x \approx 0$ and small angles — we may approximate

$$\begin{aligned} \alpha - \gamma \approx \tan(\alpha - \gamma) &= \frac{y}{d_o + x} \approx \frac{y}{d_o}, \\ \alpha + \gamma \approx \tan(\alpha + \gamma) &= \frac{y}{d_i - x} \approx \frac{y}{d_i}, \\ \gamma \approx \sin\gamma &= \frac{y}{R}. \end{aligned} \tag{4}$$

Combining these 3 equations, we have

$$\frac{y}{d_i} - \frac{y}{d_o} \approx (\alpha + \gamma) - (\alpha - \gamma) = 2\gamma \approx 2 \times \frac{y}{R} \tag{5}$$

and hence

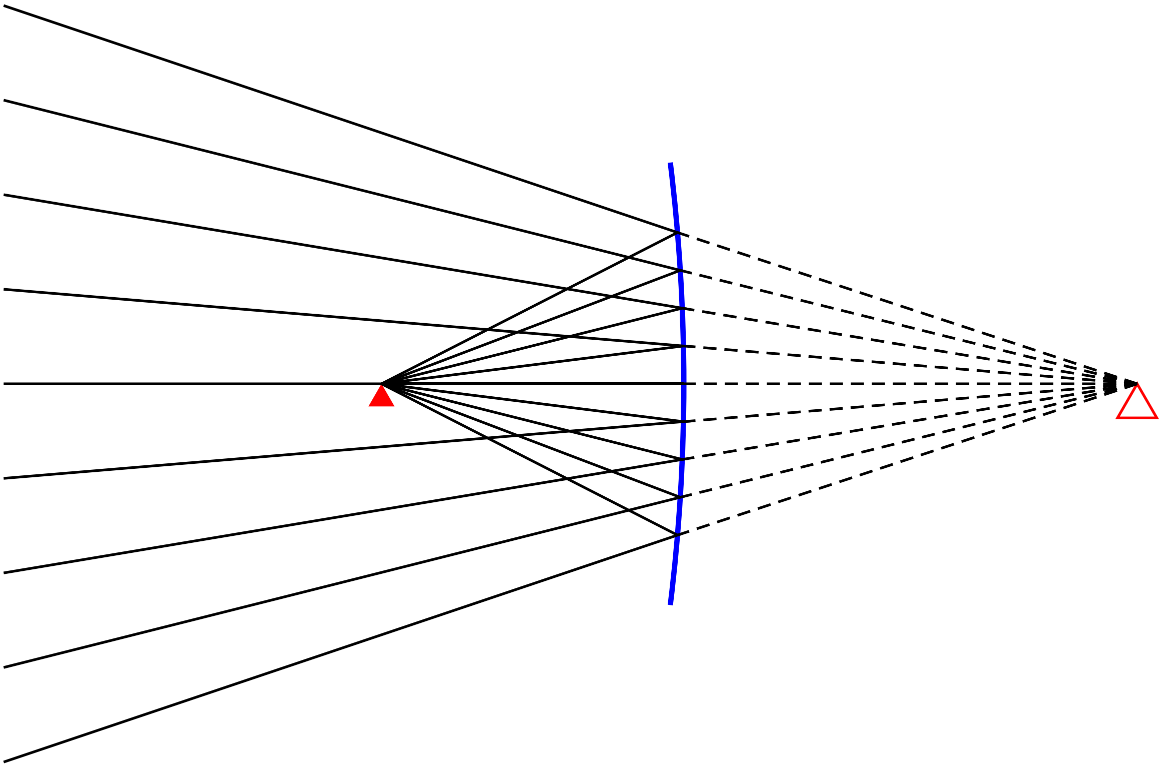
$$\frac{1}{d_i} - \frac{1}{d_r} = \frac{2}{R} \tag{6}$$

regardless of $y$. Note that according to this formula $(1/d_i) > (1/d_r)$ and hence $d_i < d_r$: *The image in a convex mirror is closer to the mirror than the original object.* Moreover, eq. (6) tells us the exact image location of any object, namely

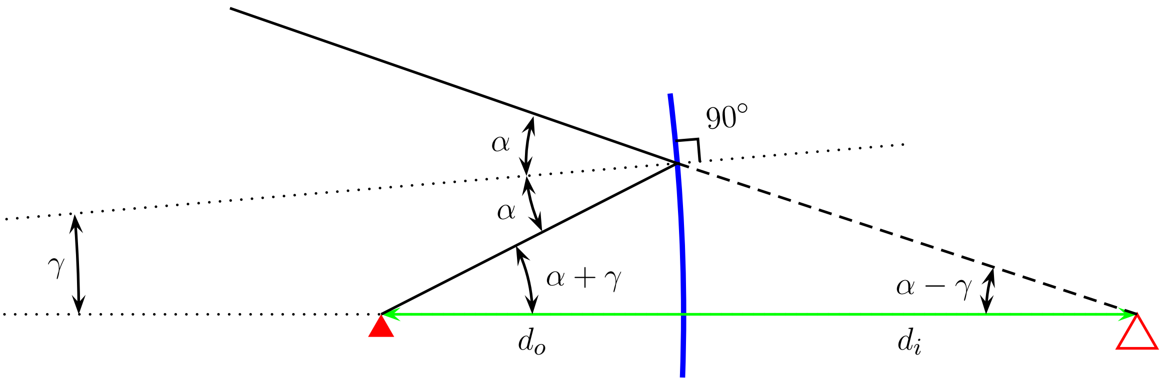
$$d_i = 1 \Big/ \left(\frac{1}{d_r} + \frac{2}{R}\right). \tag{7}$$

## Concave Mirror

For a concave mirror we have a reverse situation: the image is further away from the mirror that the object. Indeed, here is the ray diagram for a concave mirror:



And here is the detailed geometry for one particular ray




Again, $\alpha$ is the angle of incidence = angle of reflection and $\gamma$ is the angle between the mirror's surface and the $y$ axis. But now the mirror is curving in the other direction, so the angle between the incident ray and the $x$ axis is $\alpha + \gamma$ while the angle between the $x$ axis and the reflected ray is $\alpha - \gamma$. Consequently, for this ray geometry

$$\begin{aligned} \tan(\alpha+\gamma) &= \frac{y}{d_o + x}, \\ \tan(\alpha-\gamma) &= \frac{y}{d_i - x}, \\ \sin\gamma &= \frac{y}{R}, \end{aligned} \tag{8}$$

where similar to the convex case, $(x, y)$ are the coordinates of the point where the ray is reflected off the mirror and $R$ is the mirror's radius of curvature. For small $y$ — and hence $x \approx 0$ and small angles — we approximate

$$\begin{aligned} \alpha + \gamma \approx \tan(\alpha+\gamma) &= \frac{y}{d_o + x} \approx \frac{y}{d_o}, \\ \alpha - \gamma \approx \tan(\alpha-\gamma) &= \frac{y}{d_i - x} \approx \frac{y}{d_i}, \\ \gamma \approx \sin\gamma &= \frac{y}{R}. \end{aligned} \tag{9}$$

This time, combining these 3 equations gives us

$$\frac{y}{d_o} - \frac{y}{d_i} \approx (\alpha+\gamma) - (\alpha-\gamma) = 2\gamma \approx 2 \times \frac{y}{R} \tag{10}$$

and hence

$$\frac{1}{d_o} - \frac{1}{d_i} = \frac{2}{R} \tag{11}$$

regardless of $y$. Note that this formula for the concave mirror implies $(1/d_o) > (1/d_i)$ and hence $d_o < d_i$: *The image in a concave mirror is further away to the mirror than the original object.* Specifically, the image of an object at distance $d_o$ from the mirror is located at distance

$$d_i = 1 \Big/ \left(\frac{1}{d_o} - \frac{2}{R}\right). \tag{12}$$

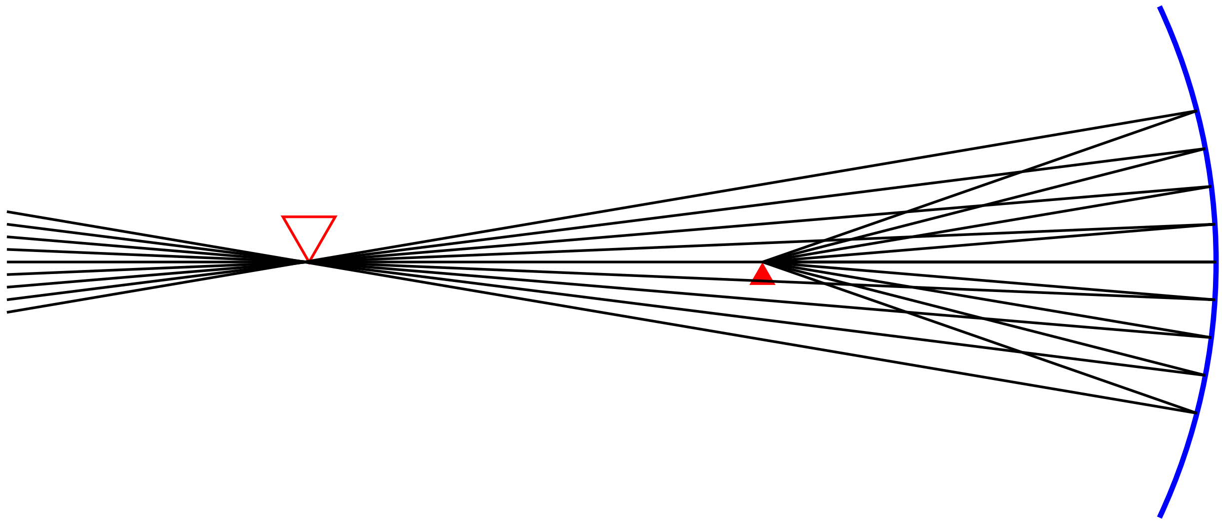
## Virtual and Real Images

Thus far, in all the ray diagrams the image was always behind the mirror. Such images are called *virtual* because the light rays do not really go behind the mirror, they just seem like they come from there. In contrast, a *real image* would appear in front of the mirror and the light rays would actually go through it. If you put a piece of paper into a real image of something very hot — the Sun, or an electric arc — the paper would burn. In comparison, a virtual image cannot burn or do anything at all to a paper you bring into it; all you can do with a virtual image is to look at it through the mirror.

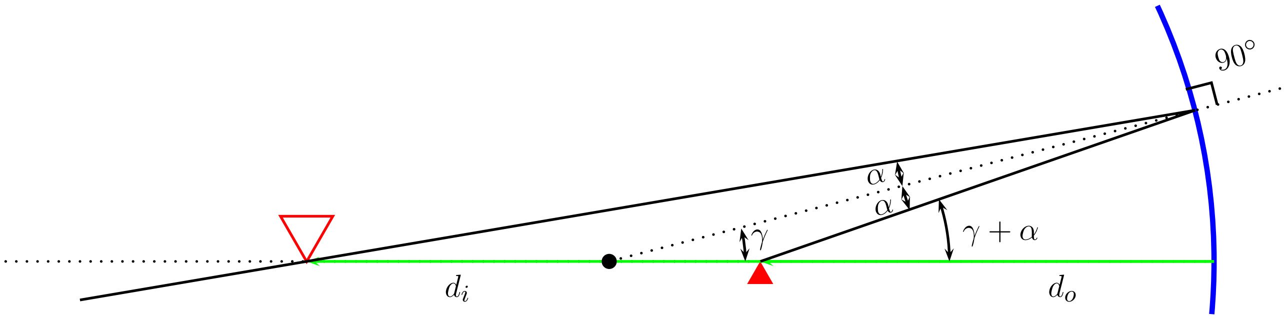
The plane and the convex mirrors can make only virtual images, but a concave mirror can make both virtual or real images. The type of the image depends on the distance between the object and the mirror: Objects at distances closer than $R/2$ have virtual images behind the mirror while objects further away than $R/2$ have real images in from of the mirror. Indeed, the diagram on the previous page shows a virtual image that obtains for positive angle $\alpha - \gamma$. Mathematically,

$$\alpha - \gamma > 0 \iff (\alpha + \gamma) > 2\gamma \iff \frac{y}{d_o} > 2 \times \frac{y}{R} \iff d_o < \tfrac{1}{2}R, \qquad (13)$$

which means that to get the virtual image we need $d_o < \frac{1}{2}R$. But when an object is further away from the mirror than $\frac{1}{2}R$ we get $\alpha - \gamma < 0$, which means that the reflected rays go back to the $x$ axis rather than away from it, and when they all cross each other at some distance $d_i$ *in front of the mirror*, they make a real image:



To find the location of the real image, let's focus on just two rays: one ray going back and forth along the $x$ axis, and one off-axis ray with non-zero $\alpha$ and $\gamma$ angles:




Applying trigonometry to this picture, we have

$$\tan(\gamma+\alpha) \;=\; \frac{y}{d_o-x} \;\approx\; \frac{y}{d_o}\,, \quad \tan(\gamma-\alpha) \;=\; \frac{y}{d_i-x} \;\approx\; \frac{y}{d_i}\,, \quad \sin\gamma \;=\; \frac{y}{R}\,. \tag{14}$$

When $y$ is small — and hence all the angles are small — we may approximate

$$\tan(\gamma+\alpha) \;\approx\; (\gamma+\alpha), \quad \tan(\gamma-\alpha) \;\approx\; (\gamma-\alpha), \quad \sin\gamma \;\approx\; \gamma. \tag{15}$$

This gives us

$$\gamma \;+\; \alpha \;\approx\; \frac{y}{d_o}\,, \quad \gamma \;-\; \alpha \;\approx\; \frac{y}{d_i}\,, \quad \gamma \;\approx\; \frac{y}{R}\,, \tag{16}$$

hence

$$\frac{y}{d_o} \;+\; \frac{y}{d_i} \;\approx\; (\gamma+\alpha) \;+\; (\gamma-\alpha) \;=\; 2\gamma \;\approx\; 2\times\frac{y}{R} \tag{17}$$

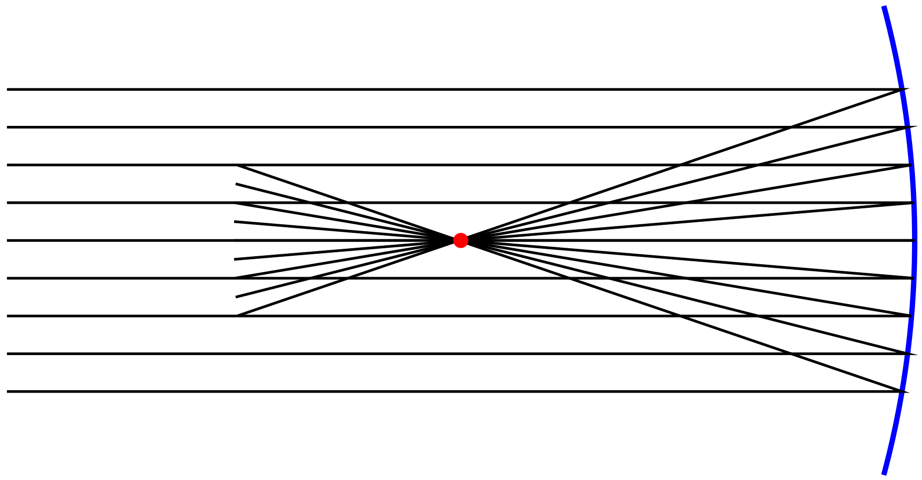
and therefore

$$\frac{1}{d_o} \;+\; \frac{1}{d_i} \;=\; \frac{2}{R}\,. \tag{18}$$

Thus, an object at distance $d_o > \frac{1}{2}R$ from a concave mirror has a real image in front of the mirror at distance

$$d_i \;=\; 1 \Bigg/ \left(\frac{2}{R} - \frac{1}{d_o}\right). \tag{19}$$

When the object gets further and further away from a concave mirror, its real image gets closer and closer. For very distant objects — such as starts — the light rays reaching the mirror are essentially parallel. When the reflect off a concave mirror, these rays intersect at the point called *the focus:*



The distance between the focus and the mirror — called the *focal distance* is simply

$$f \;=\; \frac{R}{2}\,. \tag{20}$$

If the rays come from a very hot object — like the Sun — its real image in the focus gets so hot you can burn things with it!