Gaussian Ray Tracing Technique
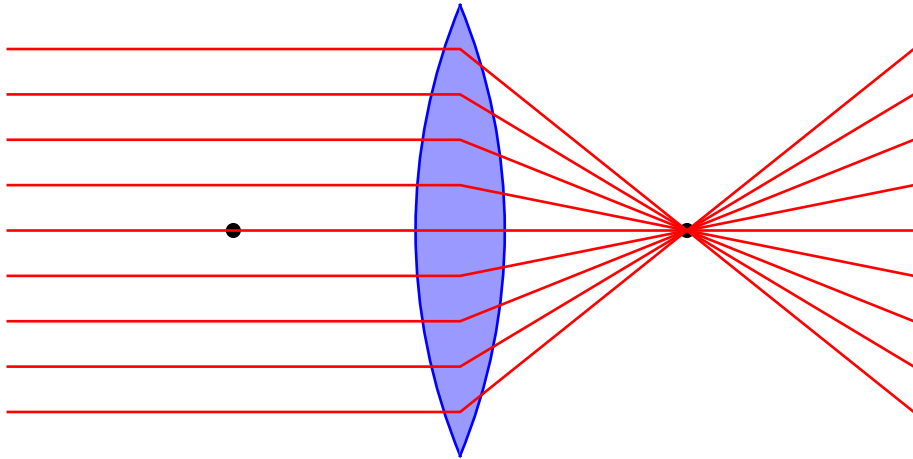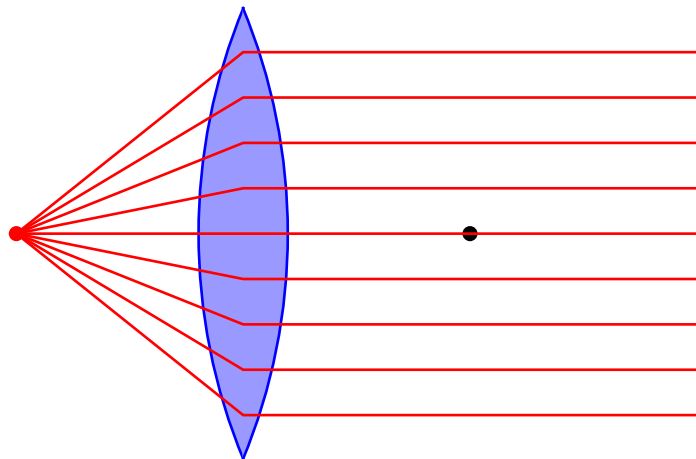
A positive lens has two *focal points* — one on each side of the lens; both are at the same *focal distance* f from the lens. Parallel rays of light coming from one side of the lens are refracted so they converge in the focus point on the other side:

On the other hand, light rays coming out of a focus are refracted so that on the other side of the lens they become parallel:

A positive lens can make both real and virtual images, depending on the distance d_o from the object to the lens: *When the distance is greater than the focal length, $d_o > f$, the image is real, t when the distance is less than the focal length, $d_o < f$, the image is virtual.*

To see how this works — and also to find the size of the image — Carl Friedrich Gauss invented a ray-tracing technique. For each interesting point of the object, we trace three rays as they go from that point to the lens and are refracted by it as shown on the following picture:
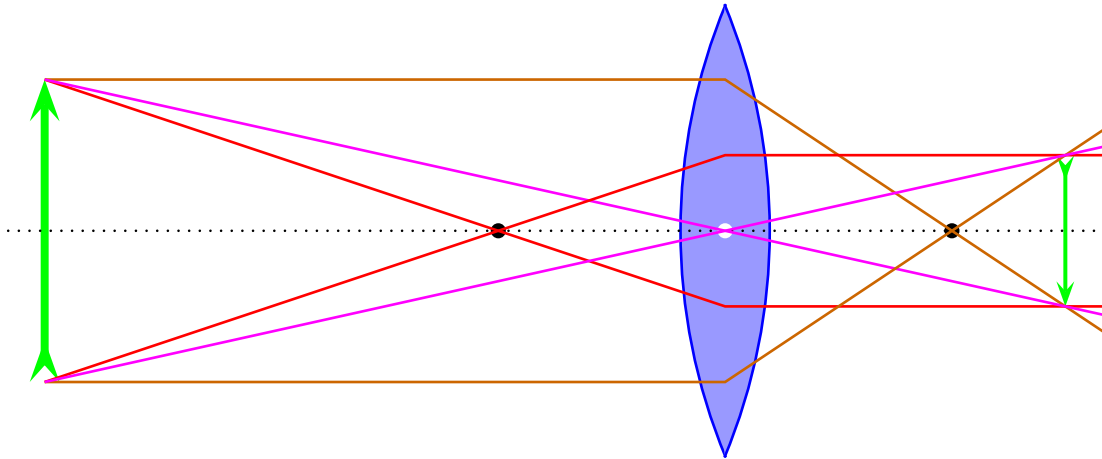
- The first ray (orange) goes parallel to the optical axis until it gets to the lens, which refracts it so it goes through the focus on the other side, and then continues in a straight line.

- The second ray (red) goes through the focus on the same side of the lens and then continues in a straight line until the lens itself. After the lens, this ray becomes parallel to the optical axis.

- Finally, the third ray (purple) goes through the center of the lens and then continues in the same direction.

Using simple Euclidean geometry, one can show that all three rays coming from the same point of the object intersect at the same point — the corresponding point of a virtual image. Its distance d_i from the lens satisfies a simple equation

$$\frac{1}{d_i} + \frac{1}{d_o} = \frac{1}{f} \quad \Longleftrightarrow \quad d_i = \frac{d_o \times f}{d_o - f} \,. \tag{1}$$

Note that the closer the object is to the focus, the more distant is the image, and vise verse.

The *magnification* of the lens is the ratio of the image size to the object size. Looking at the rays through the center of the length (purple), we can see that the magnification follows from the distances of the object and the image from the lens,

$$m \;=\; \frac{h_i}{h_o} \;=\; -\frac{d_i}{d_o}.$$ (2)

Note the minus sign here: *the real image in a positive lens is upside-down,* so we consider its magnification to be negative.

Combining equations (1) and (2), we obtain

$$m \;=\; \frac{f}{d_o - f}.$$ (3)

Note that for $d_o > 2f$, the absolute value of magnification is less than 1, while for $d_o < 2f$ it's greater than one. In other words, *the objects further away from the lens than $2f$ have reduced images, while the objects closer than $2f$ have magnified images.*

An object located between a positive lens and its focus has a virtual image instead of a real one. To see how this works, let's trace three rays for each interesting point of the object. The rays are drawn similar to the previous diagram, except that the red ray does not go through the left focal point but starts from the object as in the same direction as if it came from the left focus (dotted red line) and continues in this direction to the lens (which then refracts it parallel to the axis):
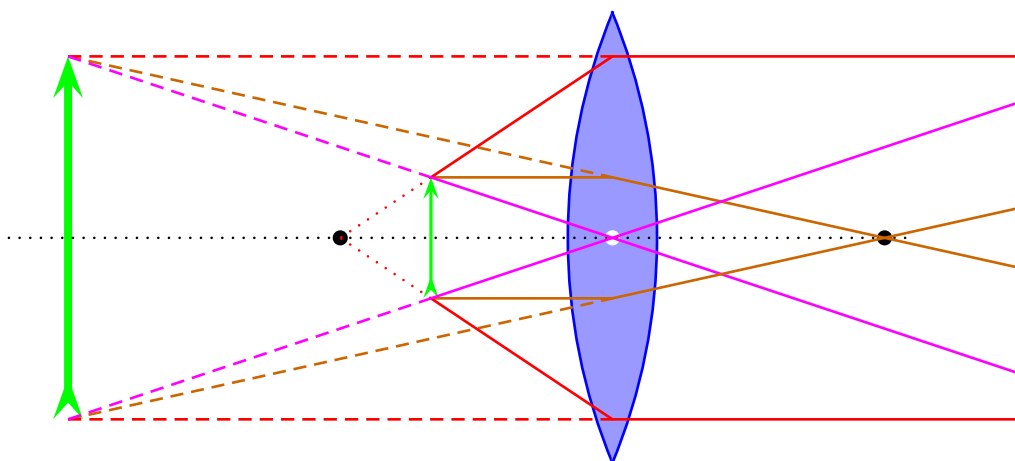
This time, the three rays from each object point do not cross each other on the right side of the lens, but if we extrapolate the refracted rays back to the left side of the lens, they would intersect at the corresponding point of the *virtual image.*

Again, Euclidean geometry gives the distance to the image

$$\frac{1}{d_o} \;-\; \frac{1}{d_i} \;=\; \frac{1}{f} \quad \Longleftrightarrow \quad d_i \;=\; \frac{d_o \times f}{f - d_o} \tag{4}$$

and the magnification

$$m \;=\; \frac{h_i}{h_o} \;=\; +\frac{d_i}{d_o} \;=\; +\frac{f}{f - d_o}\;. \tag{5}$$

Note the positive sign: Unlike the real image, *the virtual image is up-right, i.e.*, the same side up as the object. Also, a virtual image in a positive lens is always larger than the object.

NEGATIVE LENSES.

A lens whose middle is thinner than its edges is *negative.* A negative lens also has two focal points, but they are virtual: The lens makes parallel rays of light diverge as if they came from a focal point:
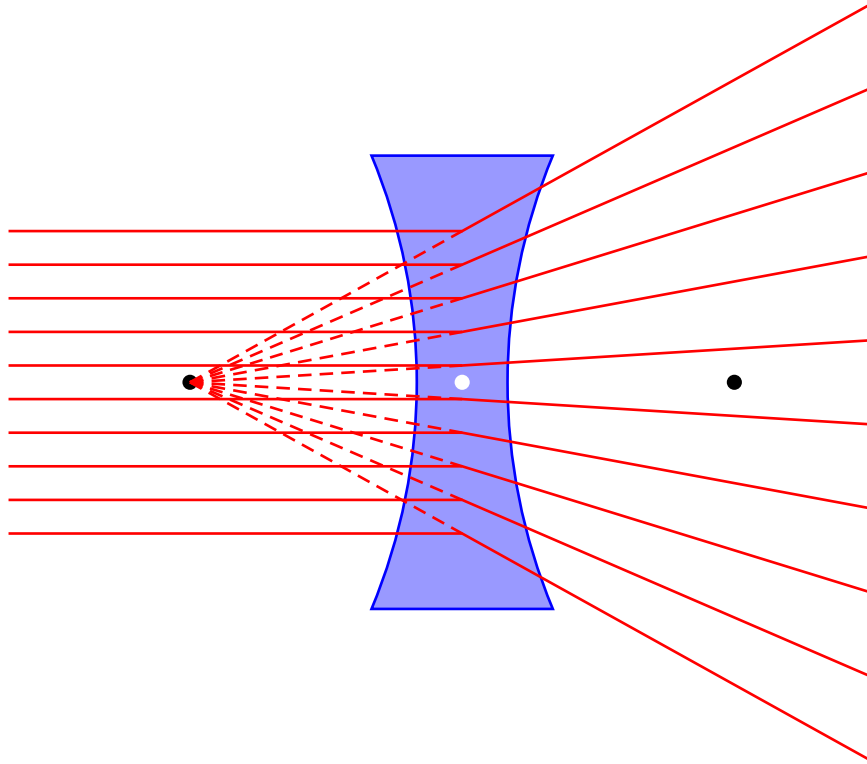

Often, the focal distance of a negative lens is written as a negative number, for example $f = -20$ cm, where the minus sign indicates that the lens is negative, while 20 cm is the distance from the lens to its virtual focus. In contrast, $f = +20$ cm would indicate a positive length with real focuses 20 cm from the lens.

The image of any object in a negative lens is always virtual, up-right, and smaller than the object itself. To see how this works, we trace three rays coming from each interesting point of the object:

- The first ray (orange) goes parallel to the axis until the lens, then turns away from the axis so that its extrapolation seems to go through the left focal point.

- The second ray (red) goes in the direction of the *right* focal point until it hits the lens, then turns parallel to the axis.

- The third ray (purple) goes through the center of the lens and continues in a straight line.
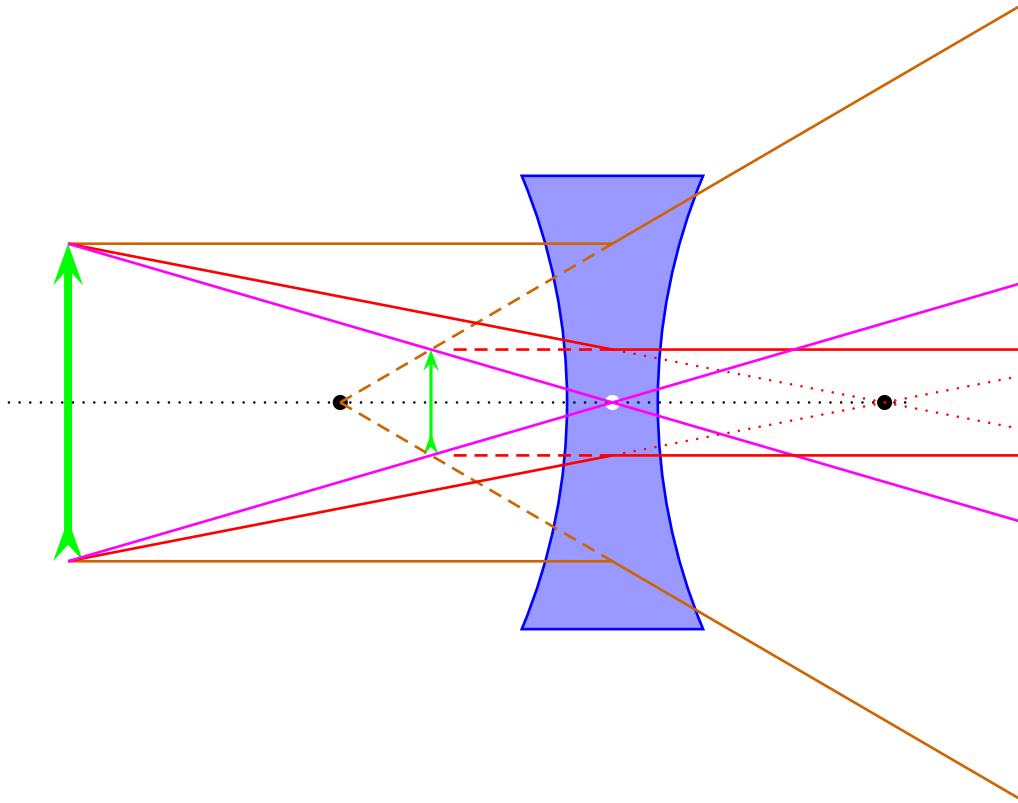
Again, Euclidean geometry tells us that the three refracted rays do not intersect at at a ream image point to the right of the lens, but their extrapolations to the left side do intersect at a virtual image point. Specifically, the virtual image is located at

$$\frac{1}{d_i} - \frac{1}{d_o} = \frac{1}{|f|} \quad \Longleftrightarrow \quad d_i = \frac{d_o \times |f|}{|f| + d_o} \tag{6}$$

and its magnification is

$$m = \frac{h_i}{h_o} = +\frac{d_i}{d_o} = +\frac{|f|}{|f| + d_o}. \tag{7}$$

Note that this magnification is always positive (up-right image) and less then 1 (the image is smaller than the object).

Similar ray-tracing techniques work for images in curved mirrors. A convex mirror of curvature radius R (*i.e.*, the mirror is a part of sphere of radius R) has focal distance $f = +\frac{1}{2}R$. Thus, the focus (black dot on the diagram below) is half-way between the mirror and its center (blue dot on the diagram). The focus is real: When parallel rays are reflected by the convex mirror, they converge and cross at this focus:
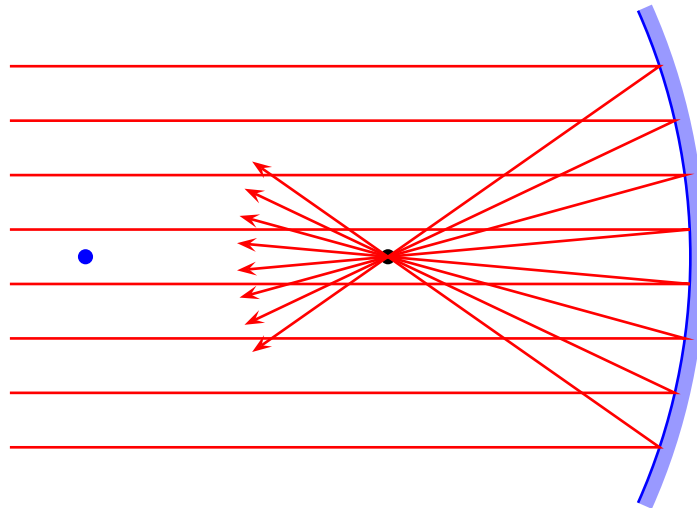
And the rays coming out of the focus become parallel after reflection:
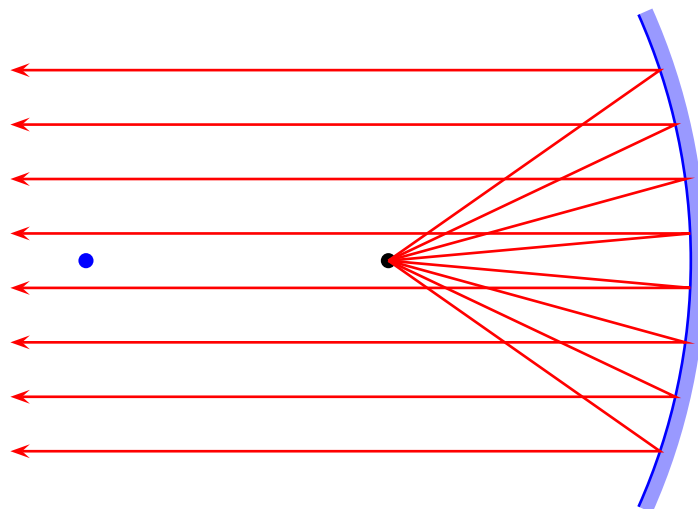
Here is a ray-tracing diagram for a real image in a convex mirror:
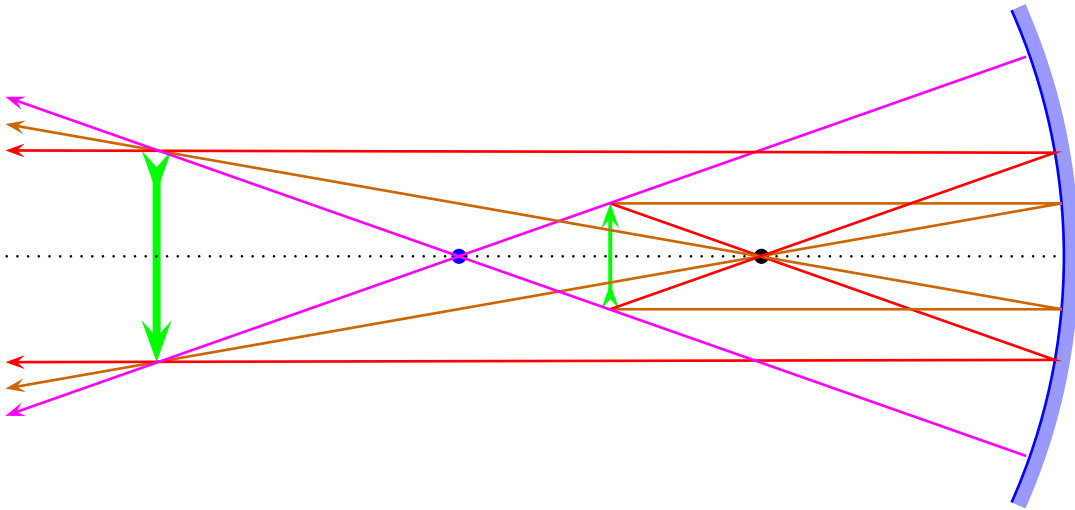
For each interesting point of an object, the first ray (orange) starts parallel to the axis, reflects of the mirror towards the focus, and then continues in a straight line. The second ray (red) goes through the focus to the mirror, the reflects back parallel to the axis. The third ray (purple) goes along the line connecting the object and the center of the mirror; this ray is reflected back towards the center. The three rays intersect at the *real image* located at d_i such that

$$\frac{1}{d_i} + \frac{1}{d_o} = \frac{1}{f} = \frac{2}{R} \iff d_i = \frac{d_o \times f}{d_o - f} . \tag{8}$$

Similar to the positive lens, the real image in a convex mirror is upside-down, and its magnification is

$$m = \frac{h_i}{h_o} = -\frac{d_i}{d_o} = -\frac{f}{d_o - f} . \tag{9}$$

Note that the objects located between the focus and the center (as on the above diagram) have enlarged images ($|m| > 1$), while the objects located further away then the center have reduces images ($|m| < 1$).

Objects closer to a convex mirror than its focus have virtual rather than real images:
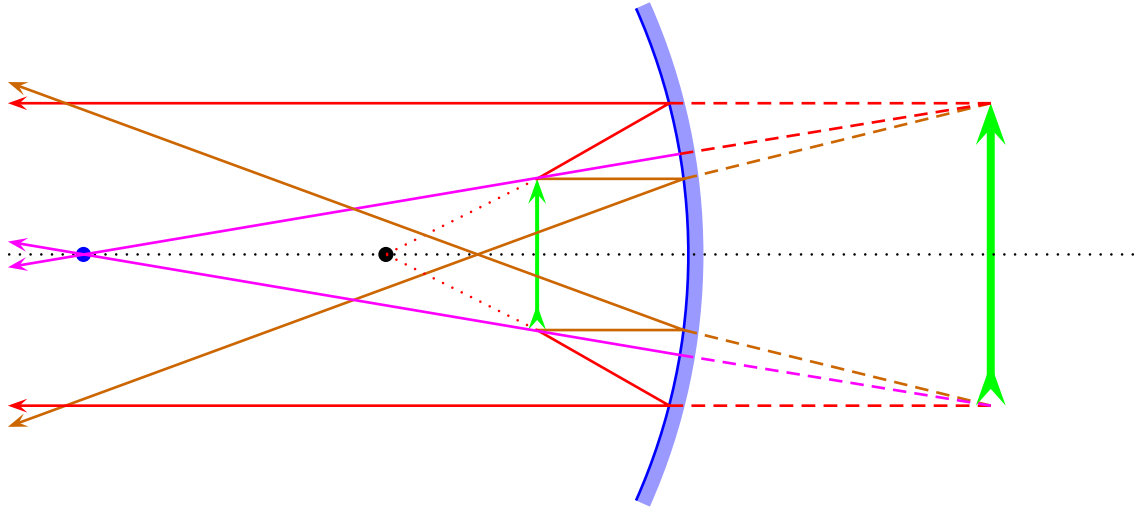
Similar to virtual images in positive lenses, the virtual image in a convex mirror is up-right, its distance d_i behind the mirror is given by

$$\frac{1}{d_i} = \frac{1}{f} - \frac{1}{d_o} \quad \Longleftrightarrow \quad d_i = \frac{d_o \times f}{f - d_o}, \tag{10}$$
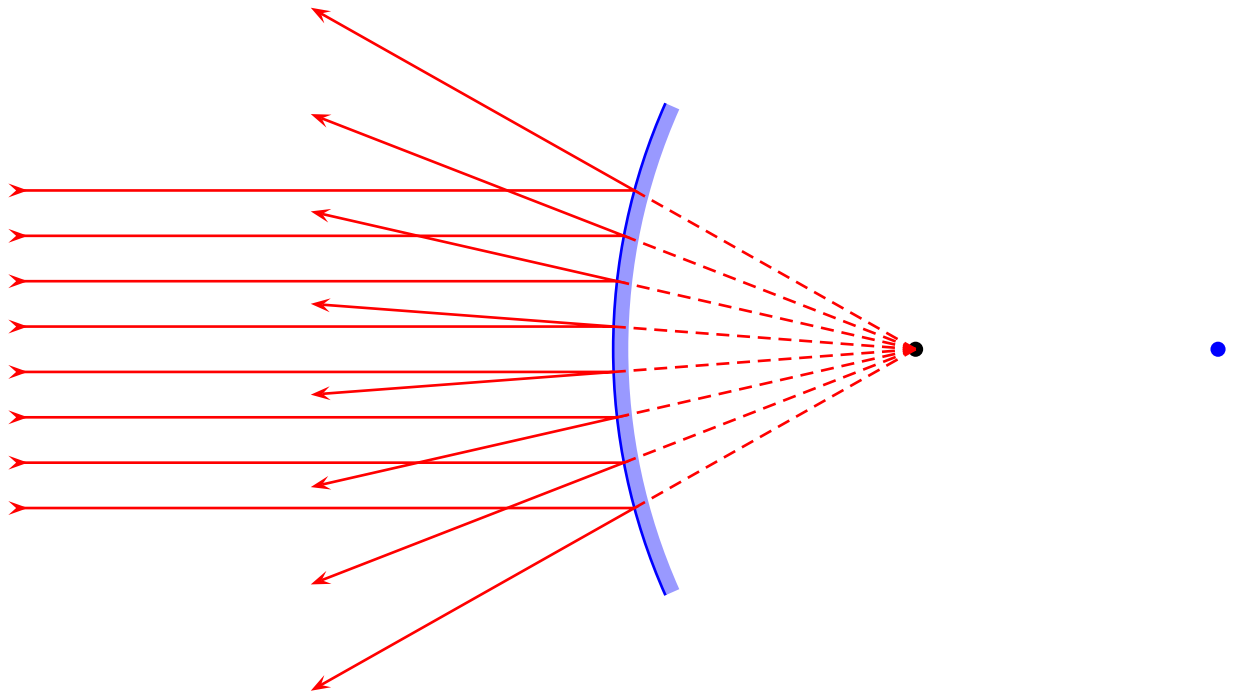
and the magnification is

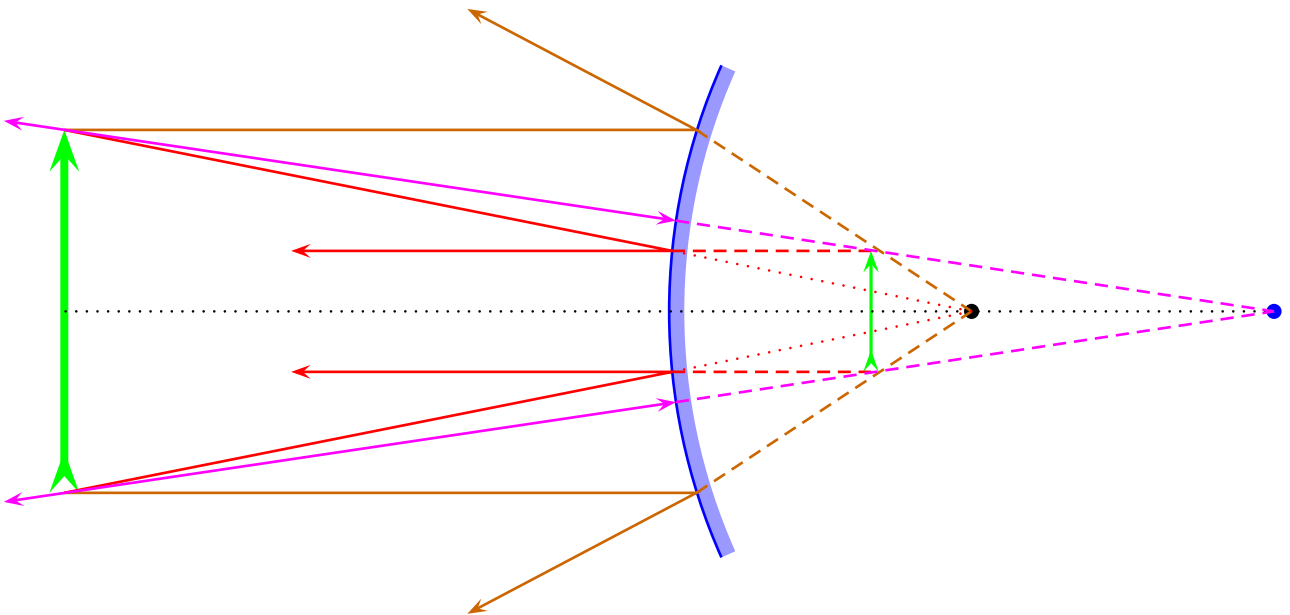$$m = \frac{h_i}{h_o} = +\frac{d_i}{d_o} = +\frac{f}{f - d_o}. \tag{11}$$

Note that this magnification is always greater then 1, *i.e.*, the virtual image in a convex mirror is always larger than the object.

Finally, a *convex mirror* has focal length $f = -\frac{1}{2}R$, where the minus sign indicates that the focus is virtual: When parallel rays are reflected from a convex mirror, they diverge and

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seem to come from the virtual focus behind the mirror:

All images in convex mirrors are virtual, up-right, and smaller that the object. Here is the ray diagram:

Specifically, the distance to the virtual image behind the mirror is

$$\frac{1}{d_i} = \frac{1}{|f|} + \frac{1}{d_o} \iff d_i = \frac{d_o \times |f|}{|f| + d_o}, \tag{12}$$

and the magnification is

$$m = \frac{h_i}{h_o} = +\frac{d_i}{d_o} = +\frac{|f|}{|f| + d_o} < 1. \tag{13}$$