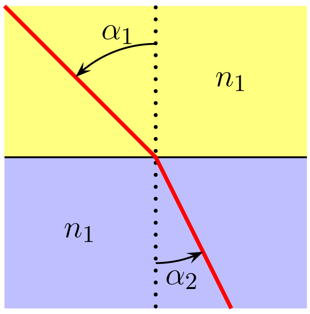
Problem **#1:**

When a light ray goes from one transparent medium to another — for example, from air to glass, or from glass to air — its direction changes according to the *Snell's Law of Refraction*

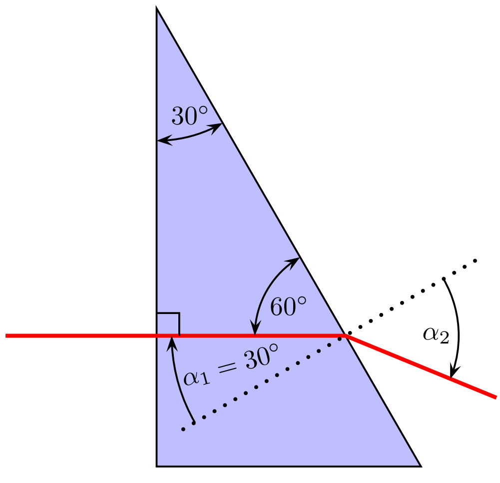


$$n_1 \times \sin \alpha_1 \;=\; n_2 \times \sin \alpha_2 \,. \tag{1}$$

Here n_1 and n_2 are the refraction indices of the two media, α_1 is the angle of incidence, α_2 is the angle of refraction, and both angles are counted from the perpendicular to the boundary.

The light ray in question crosses two interfaces, first from the air to the glass, and then from the glass back to the air. At the first interface, the ray comes in $\perp$ to the air-glass boundary — *i.e.*, at zero incidence angle. Consequently, the angle of refraction must also be zero, which means that the ray continues into the glass along the same direction — $\perp$ to the boundary.

In a prism, the second air-glass boundary is tilted relative to the first boundary; for the prism in question, the tilt angle is 30°. Consequently, the ray approaches the second interface at incidence angle $\alpha_1 = 30°$ as shown on the following diagram:



This time, the ray does change direction. To find the angle of refraction α_2, we use the Snell's Law (1) for refraction from the glass into the air, thus

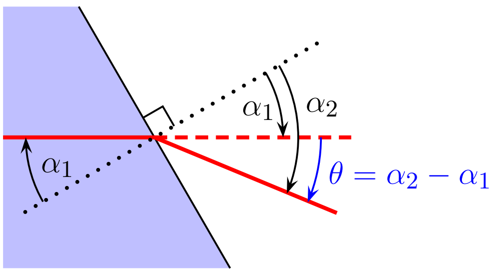
$$n_1 = n_{\text{glass}} = 1.6 \quad \text{and} \quad n_2 = n_{\text{air}} \approx 1. \tag{2}$$

Hence,

$$\sin\alpha_2 = \frac{n_1}{n_2} \times \sin\alpha_1 = \frac{1.6}{1} \times \sin 30^\circ = 0.8 \tag{3}$$

and $\alpha_1 = \arcsin(0.8) \approx 53^\circ$.

Note that since the ray goes from the glass to the air we have $n_2 < n_1$ and consequently $\alpha_2 > \alpha_1$ — the refraction angle is large than the incidence angle. This means that the light ray bends away from the perpendicular, *i.e.*, *the ray bends down.* The bending angle θ is the difference $\alpha_2 - \alpha_1$, which should be obvious from the following diagram:



Numerically,

$$\theta = \alpha_2 - \alpha_1 = 53^\circ - 30^\circ = 23^\circ. \tag{4}$$

So here is the bottom line: The light ray bends 23° *down.*

Problem **#2:**

(a) Note: the image is *behind* the lens, *i.e.*, on the same side of the lens as the object, so this image must be virtual rather than real. The distance to a virtual image depends on a type of lens: Positive lenses make virtual images more distant then their objects,

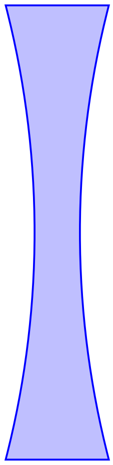
$$\frac{1}{d_i} = \frac{1}{d_o} - \frac{1}{f} < \frac{1}{d_o} \quad \Longrightarrow \quad d_i > d_o\,, \tag{5}$$

while negative lenses make virtual image closer then the objects,

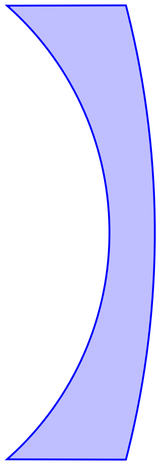
$$\frac{1}{d_i} = \frac{1}{d_o} + \frac{1}{|f|} > \frac{1}{d_o} \quad \Longrightarrow \quad d_i < d_o\,. \tag{6}$$

The virtual image in question is closer to the lens than the object — $(d_i = 20\text{ cm}) < (d_o = 30\text{ cm})$ — so the lens must be *negative.*

A negative lens is thinner in the middle than near the edges, for example



or



(b) The focal length of the negative lens follows from eq. (6) and known distances d_o to the object and d_i to the image:

$$\begin{aligned}
\frac{1}{d_i} &= \frac{1}{d_o} + \frac{1}{|f|}\,, \\
\frac{1}{|f|} &= \frac{1}{d_i} - \frac{1}{d_o}\,, \\
|f| &= \frac{d_i \times d_o}{d_o - d_i} \\
&= \frac{(20\text{ cm}) \times (30\text{ cm})}{(30\text{ cm}) - (20\text{ cm})} \\
&= 60\text{ cm}.
\end{aligned} \tag{7}$$

By convention, focal length of a negative lens is written as a negative number, so the lens in question has focal length $f = -60$ cm.

However, I will not take any points off for ignoring this convention. Students that have answered this question as simply $f = 60$ cm — without the sign — will get full credit for this part of the problem.

(c) Since the ray that goes through the center of the lens is not refracted, the magnification of a virtual image is simply

$$m \stackrel{\text{def}}{=} \frac{h_i}{h_o} = \frac{d_i}{d_o}. \tag{8}$$

In particular, the image in question has magnification

$$m = \frac{d_i}{d_o} = \frac{20 \text{ cm}}{30 \text{ cm}} = \frac{2}{3}. \tag{9}$$

Problem **#3:**

In any radioactive decay, the fraction of atoms surviving after time t decreases exponentially as

$$\frac{N(t)}{N_0} = 2^{-t/T} \tag{10}$$

where T is the half-life time of the isotope in question. Note that different isotopes have very different half-life times, which range from microseconds to billions of years. In particular, the half-life time of uranium-235 is 6 times shorter than that of uranium-238.

(a) The age of the Earth — 4500 million years — happens to be close to the half-life time of the ^{238}U isotope. During this time, $\frac{1}{2}$ of the original ^{238}U atoms have decayed while the other $\frac{1}{2}$ is still with us today. This means that for every ^{238}U atom the Earth has today, there were **2** such atoms when the Earth was formed.

(b) The ^{235}U isotope has a shorter half-life time $T = 750$ million years, or 1/6 of the Earth's age $t = 4500$ million years. Hence, the fraction of original ^{235}U atoms that survive till today is only

$$\frac{N(t)}{N_0} = 2^{-t/T} = 2^{-6} = \frac{1}{2^6} = \frac{1}{64}. \tag{11}$$

In other words, only 1 in 64 original ^{235}U atoms has survived till today, which means that for each ^{235}U atom the Earth has now, it had 64 such atoms when it was formed.

(c) In parts (a) and (b) we saw that the surviving fractions of ${}^{235}\mathrm{U}$ and ${}^{238}\mathrm{U}$ are rather different:

$$N_{\text{today}}\left({}^{238}\mathrm{U}\right) \;=\; \frac{1}{2}\,N_{\oplus\,\text{formation}}\left({}^{238}\mathrm{U}\right) \quad \text{while} \quad N_{\text{today}}\left({}^{235}\mathrm{U}\right) \;=\; \frac{1}{64}\,N_{\oplus\,\text{formation}}\left({}^{235}\mathrm{U}\right). \tag{12}$$

Consequently, the isotope ratio today is quite different than when the Earth was formed:

$$\frac{N_{\text{today}}\left({}^{235}\mathrm{U}\right)}{N_{\text{today}}\left({}^{238}\mathrm{U}\right)} \;=\; \frac{\frac{1}{64}N_{\oplus\,\text{formation}}\left({}^{235}\mathrm{U}\right)}{\frac{1}{2}N_{\oplus\,\text{formation}}\left({}^{238}\mathrm{U}\right)} \;=\; \frac{2}{64}\times\frac{N_{\oplus\,\text{formation}}\left({}^{235}\mathrm{U}\right)}{N_{\oplus\,\text{formation}}\left({}^{238}\mathrm{U}\right)}\,. \tag{13}$$

Today,

$$\frac{N_{\text{today}}\left({}^{235}\mathrm{U}\right)}{N_{\text{today}}\left({}^{238}\mathrm{U}\right)} \;=\; \frac{1}{138}\,, \tag{14}$$

but when the Earth was formed 4500 million years ago, the ratio was

$$\frac{N_{\oplus\,\text{formation}}\left({}^{235}\mathrm{U}\right)}{N_{\oplus\,\text{formation}}\left({}^{238}\mathrm{U}\right)} \;=\; \frac{64}{2}\times\frac{N_{\text{today}}\left({}^{235}\mathrm{U}\right)}{N_{\text{today}}\left({}^{238}\mathrm{U}\right)} \;=\; \frac{64}{276}\,, \tag{15}$$

i.e., for every 64 atoms of ${}^{235}\mathrm{U}$ there were 280 atoms of ${}^{238}\mathrm{U}$.

In terms of isotope fractions of natural uranium as it existed 4500 million years ago,

$$\begin{aligned} \text{fraction}\left({}^{235}\mathrm{U}\right) \;&=\; \frac{64}{64+276} \;=\; 18.8\%,\\ \text{fraction}\left({}^{238}\mathrm{U}\right) \;&=\; \frac{280}{64+276} \;=\; 81.2\%. \end{aligned} \tag{16}$$

PS: The half-life times used in this problem were rounded to simplify your calculations. In reality, ^{238}U has half-life time $T_{238} = 4468$ million years, while ^{235}U has half-life time $T_{235} = 704$ million years. Also, the age of Earth is about $t = 4540$ million years.

For these more accurate numbers, $t/T_{238} = 1.016$, so the surviving fraction of the original ^{238}U atoms is

$$\frac{N_{\text{today}}\left({}^{238}\text{U}\right)}{N_{\oplus\,\text{formation}}\left({}^{238}\text{U}\right)} = 2^{-t/T_{238}} = 0.4944 = \frac{1}{2.022}\,. \tag{17}$$

Likewise, $t/T_{235} = 6.449$, so the surviving fraction of the original ^{235}U atoms is

$$\frac{N_{\text{today}}\left({}^{235}\text{U}\right)}{N_{\oplus\,\text{formation}}\left({}^{235}\text{U}\right)} = 2^{-t/T_{235}} = 0.011447 = \frac{1}{87.36}\,. \tag{18}$$

Finally, the uranium isotope ratio when the Earth was formed was

$$\frac{N_{\oplus\,\text{formation}}\left({}^{235}\text{U}\right)}{N_{\oplus\,\text{formation}}\left({}^{238}\text{U}\right)} = \frac{87.36}{2.022} \times \frac{N_{\text{today}}\left({}^{235}\text{U}\right)}{N_{\text{today}}\left({}^{238}\text{U}\right)} = 0.313 \approx \frac{5}{16}\,, \tag{19}$$

i.e., for each 5 atoms of ^{235}U there were 16 atoms of ^{238}U. In terms of isotope fractions of natural uranium as it was when the Earth was formed,

$$\begin{aligned} \text{fraction}\left({}^{235}\text{U}\right) &= \frac{5}{5+16} = 23.9\%, \\ \text{fraction}\left({}^{238}\text{U}\right) &= \frac{16}{5+16} = 76.1\%. \end{aligned} \tag{20}$$

Problem **#4:**

(a) The decay chain (1) spells out the atomic numbers of all the isotopes but not their mass numbers. The atomic number Z is the number of protons in the nucleus and also the number of electrons in the neutral atom. In an α decay Z decreases by 2 because an α particle carries away 2 of the protons. In a β decay, Z increases by 1 because one of the neurons turns into a proton (plus an electron and anti-neutrino that fly away). In a γ decay Z does not change. This gives us a simple rule for distinguishing different decay types: Take the difference ΔZ between the mother and daughter nuclei and look it up in the following table:

$$\begin{aligned} \Delta Z = -2: &\quad \alpha \text{ decay,} \\ \Delta Z = +1: &\quad \beta \text{ decay,} \\ \Delta Z = 0: &\quad \gamma \text{ decay,} \\ \text{other } \Delta Z: &\quad \text{other type of decay, not studied in this class.} \end{aligned} \tag{21}$$

Thus, for the decay chain in question, we have

$$\begin{array}{rcllll}
{}^{235}_{92}\mathrm{U} & \to & {}^{???}_{90}\mathrm{Th} & \Delta Z & = -2: & \alpha \text{ decay,} \\
{}^{???}_{90}\mathrm{Th} & \to & {}^{???}_{91}\mathrm{Pa} & \Delta Z & = +1: & \beta \text{ decay,} \\
{}^{???}_{91}\mathrm{Pa} & \to & {}^{???}_{89}\mathrm{Ac} & \Delta Z & = -2: & \alpha \text{ decay,} \\
{}^{???}_{89}\mathrm{Ac} & \to & {}^{???}_{90}\mathrm{Th} & \Delta Z & = +1: & \beta \text{ decay,} \\
{}^{???}_{90}\mathrm{Th} & \to & {}^{???}_{88}\mathrm{Ra} & \Delta Z & = -2: & \alpha \text{ decay,} \\
{}^{???}_{88}\mathrm{Ra} & \to & {}^{???}_{86}\mathrm{Rn} & \Delta Z & = -2: & \alpha \text{ decay,} \\
{}^{???}_{86}\mathrm{Rn} & \to & {}^{???}_{84}\mathrm{Po} & \Delta Z & = -2: & \alpha \text{ decay,} \\
{}^{???}_{84}\mathrm{Po} & \to & {}^{???}_{82}\mathrm{Pb} & \Delta Z & = -2: & \alpha \text{ decay,} \\
{}^{???}_{82}\mathrm{Pb} & \to & {}^{???}_{83}\mathrm{Bi} & \Delta Z & = +1: & \beta \text{ decay,} \\
{}^{???}_{83}\mathrm{Bi} & \to & {}^{???}_{81}\mathrm{Tl} & \Delta Z & = -2: & \alpha \text{ decay,} \\
{}^{???}_{81}\mathrm{Tl} & \to & {}^{207}_{82}\mathrm{Pb} & \Delta Z & = +1: & \beta \text{ decay.}
\end{array}$$

(b) An α decay decreases the mass number A of an isotope by 4 because the α particle carries away 4 nucleons (2 protons and 2 neutrons). A β decay does not change the mass number: while a neutron turns into a proton, the net number of nucleons does not change. Likewise, a γ decay does not change the mass number of the isotope.

For the decay chain in question, we know the initial mass number, and we have found (in part (a)) which decays are α and which are β. To find the mass numbers of all the isotopes involves, we simply follows the chain: for an α decay A decreases by 4 while for a β decay

A stays unchanged. In this way, we obtain

$$\begin{aligned} {}^{235}_{92}\mathrm{U} &\xrightarrow{\alpha} {}^{231}_{90}\mathrm{Th} \xrightarrow{\beta} {}^{231}_{91}\mathrm{Pa} \xrightarrow{\alpha} {}^{227}_{89}\mathrm{Ac} \xrightarrow{\beta} {}^{227}_{90}\mathrm{Th} \xrightarrow{\alpha} {}^{223}_{88}\mathrm{Ra} \xrightarrow{\alpha} \\ &\xrightarrow{\alpha} {}^{219}_{86}\mathrm{Rn} \xrightarrow{\alpha} {}^{215}_{84}\mathrm{Po} \xrightarrow{\alpha} {}^{211}_{82}\mathrm{Pb} \xrightarrow{\beta} {}^{211}_{83}\mathrm{Bi} \xrightarrow{\alpha} {}^{207}_{81}\mathrm{Tl} \xrightarrow{\beta} {}^{207}_{82}\mathrm{Pb}. \end{aligned} \tag{22}$$

Problem **#5:**

The positron and the electron have net mass $2m_e$. When they annihilate, all of this mass is converted to energy

$$E = 2m_e \times c^2 = 2(9.11 \cdot 10^{-31}\ \mathrm{kg}) \times (3.00 \cdot 10^8\ \mathrm{m/s}) = 1.640 \cdot 10^{-13}\ \mathrm{J}. \tag{23}$$

This energy is carried away by two photons. Assuming equal division of net energy between the photons,[⋆] each photons has

$$E_\gamma = \tfrac{1}{2} E_{\mathrm{net}}\ m_e c^2 = 8.20 \cdot 10^{-14}\ \mathrm{J}. \tag{24}$$

According to Planck's formula, photon energy is related to the electromagnetic wave's frequency f as $E_\gamma = h \times f$ where h is the Planck's constant. For the energy (24), the frequency is

$$f = \frac{E_\gamma}{h} = \frac{8.20\ \cdot 10^{-14}\ \mathrm{J}}{6.63 \cdot 10^{-34}\ \mathrm{J \cdot s}} = 1.24 \cdot 10^{20}\ \mathrm{Hz} \tag{25}$$

and hence the wavelength is

$$\lambda = \frac{c}{f} = \frac{3.00\ \cdot 10^8\ \mathrm{m/s}}{1.24 \cdot 10^{20}\ \mathrm{Hz}} = 2.42 \cdot 10^{-12}\ \mathrm{m}. \tag{26}$$

⋆ Actually, the 2 final photons always have equal energies when the initial positron and electron are at rest. Indeed, by momentum conservation

$$\vec{p}(\gamma\#1) + \vec{p}(\gamma\#2) = \vec{p}(e^+) + \vec{p}(e^-) = \vec{0},$$

which means that the two photons should have equal but opposite momenta. But the momentum of a photon is related to its energy as $E = |\vec{p}| \times c$, so equal momenta of the two photons imply equal energies.