

Rotation Kinematics, Moment of Inertia, and Torque

Mathematically, rotation of a rigid body about a fixed axis is analogous to a linear motion in one dimension. Although the physical quantities involved in rotation are quite distinct from their counterparts for the linear motion, the formulae look very similar and may be manipulated in similar ways. Here is the correspondence table:

Linear Motion	Angular Motion
Linear displacement Δx	Angular displacement $\Delta\varphi$
Linear velocity $v = \frac{\Delta x}{\Delta t}$	Angular velocity $\omega = \frac{\Delta\varphi}{\Delta t}$
Linear acceleration $a = \frac{\Delta v}{\Delta t}$	Angular acceleration $\alpha = \frac{\Delta\omega}{\Delta t}$
Motion with constant acceleration: $v(t) = v_0 + at, \quad x(t) = x_0 + v_0t + \frac{1}{2}at^2$	Rotation with constant angular acceleration: $\omega(t) = \omega_0 + \alpha t, \quad \varphi(t) = \varphi_0 + \omega_0t + \frac{1}{2}\alpha t^2$
Mass m	Moment of inertia I
Kinetic energy $K = \frac{1}{2}mv^2$	Kinetic energy $K = \frac{1}{2}I\omega^2$
Force F	Torque τ
Equation of Motion $F = ma$	Equation of Motion $\tau = I\alpha$
Work $W = F\Delta x$	Work $W = \tau\Delta\varphi$
Linear momentum $P = mv$	Angular Momentum $L = I\omega$

Now consider the motion of some small part of the body located at distance r from the axis of rotation. (If we let the z axis run along the axis of rotation, $r^2 = x^2 + y^2 + \cancel{z^2}$.) As the body rotates through angle $\Delta\varphi$, the part moves along a circular arc of radius r and length $L = r \times \Delta\varphi$ (for $\Delta\varphi$ in radians). Consequently, the linear velocity of the part is related to the angular velocity of the body as

$$v = r \times \omega. \tag{1}$$

The direction of the velocity vector \vec{v} in 3D is tangent to the circle and therefore perpendicular to the radius vector and also to the axis of rotation.

As to the linear acceleration of the part, it comprises two terms,

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}_t + \vec{\mathbf{a}}_c \quad (2)$$

the tangential acceleration and the centripetal acceleration. The tangential acceleration has magnitude

$$a_t = r \times \alpha \quad (3)$$

and direction parallel to the velocity vector $\vec{\mathbf{v}}$. The centripetal acceleration has magnitude

$$a_c = r \times \omega^2 = \frac{v^2}{r} \quad (4)$$

(regardless of α) and direction towards the axis of rotation. Since the tangential and the centripetal accelerations are always perpendicular to each other, the net acceleration vector $\vec{\mathbf{a}}$ has magnitude

$$a = \sqrt{a_t^2 + a_c^2} = r \times \sqrt{\alpha^2 + \omega^4}. \quad (5)$$

The net kinetic energy of a rotating body is simply the sum of kinetic energies of all its parts. Thus,

$$\begin{aligned} K &= \sum_i^{\text{parts}} \frac{1}{2} m_i \times v_i^2 \\ &= \sum_i^{\text{parts}} \frac{1}{2} m_i \times (r_i \times \omega)^2 \\ &= \frac{\omega^2}{2} \times \sum_i^{\text{parts}} m_i \times r_i^2 \\ &= \frac{\omega^2}{2} \times I \end{aligned} \quad (6)$$

where

$$I = \sum_i^{\text{parts}} m_i \times r_i^2 \equiv \sum_i^{\text{parts}} m_i \times (x_i^2 + y_i^2 + \cancel{x_i^2}) \quad (7)$$

is the **moment of inertia** of the body — the rotational analogue of the mass.

The moment of inertia of a body depends on its mass, size, and shape, and also on a particular axis around which the body is rotated. Here are a few examples:

- Solid rod of length L and uniform density; axis \perp to the rod. For axis through the end of the rod, $I = \frac{1}{3}ML^2$. For axis through the middle of the rod, $I = \frac{1}{12}ML^2$.
- A thin cylindrical shell of radius R and uniform density. For axis=cylinder's axis, $I = MR^2$. Same formula for a thin hoop and an axis through the center, \perp to the hoops' plane. But for axis=diameter of the hoop, $I = \frac{1}{2}MR^2$.
- A solid cylinder or disk of radius R and uniform density. For axis=cylinder's axis, $I = \frac{1}{2}MR^2$.
- A thin spherical shell of radius R and uniform density and thickness. For any axis through the sphere's center, $I = \frac{2}{3}MR^2$.
- A solid ball of radius R and uniform density. For any axis through the sphere's center, $I = \frac{2}{5}MR^2$.
- Any body whose mass is concentrated in several points (or rather parts of relatively small sizes) — use eq. (7).

The rotational analogue of the Newton's Second Law $ma = F$ is

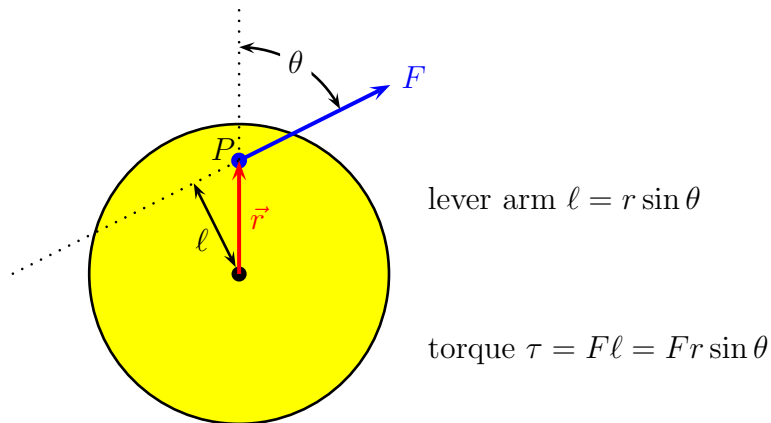
$$I\alpha = \tau \tag{8}$$

where τ is the **torque** — the rotational analogue of the force. The torque of a force $\vec{\mathbf{F}}$ depends on the force's magnitude and direction, and also on location of the point where the force is applied. It is defined as a product of a force (acting on a rotating body) and its **lever arm**,

$$\tau = F \times \ell, \tag{9}$$

where the lever arm ℓ is the distance between the axis of rotation and the line of force — line through the point P where the force acts in the direction of the force vector $\vec{\mathbf{F}}$. In a 2D plane

perpendicular to the axis of rotation,



In components, force $\vec{\mathbf{F}} = (F_x, F_y, F_z)$ acting at point $\vec{\mathbf{r}} = (x, y, z)$ has torque (around the z axis)

$$\tau = xF_y - yF_x. \quad (10)$$

Note that the F_z component of the force — parallel to the axis — does not contribute to the torque.

When a body does not have a fixed axis of rotation but can turn in all three dimensions, we need to generalize the torque to a vector $\vec{\tau}$. The torque of a force $\vec{\mathbf{F}}$ acting at point $\vec{\mathbf{r}}$ is the *vector product*

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}. \quad (11)$$

In components,

$$\begin{aligned} \tau_x &= yF_z - zF_y, \\ \tau_y &= zF_x - xF_z, \\ \tau_z &= xF_y - yF_x. \end{aligned} \quad (12)$$

But in this class we shall limit ourselves to torques around a fixed axis.