Problem **1**:

The forces acting on the car comprise its weight mg, the normal force N from the road that cancels it, and the *static friction force* f that provides for the horizontal acceleration of the car, $f = ma$. When moves along a curve, it has a sideways normal acceleration

$$a = \frac{v^2}{R} \qquad \text{(S.1)}$$

where R is the curvature radius and v is the car's speed. To provide this acceleration, the car needs static friction force

$$f = ma = \frac{mv^2}{R}. \qquad \text{(S.2)}$$

But the static friction force is limited by the normal force,

$$f \leq \mu_s \times N = \mu_s \times mg. \qquad \text{(S.3)}$$

Consequently, the car must have

$$\frac{mv^2}{R} = f \leq \mu_s mg \qquad \text{(S.4)}$$

and hence

$$v \leq \sqrt{\mu_s g \times R} \qquad \text{(S.5)}$$

regardless of the car's mass m. Numerically, for the road curve in question,

$$v \leq v_{\text{max}} = \sqrt{0.8 \times 9.8\ \text{m/s}^2 \times 51\ \text{m}} = 20\ \text{m/s} = 45\ \text{MPH}. \qquad \text{(S.6)}$$

If the car tries to go through this curve at higher speed, it would need a higher sideways acceleration $a = v^2/R$ than the maximum $\mu_s \times g$ that the static friction force can provide. Consequently, the car would skid sideways rather than follow the curving road; a few seconds later, it would end up in a ditch or worse.

Problem # **2**:

The angular velocity – and hence the speed and the orbital period — of a satellite in a circular orbit of radius R around a planet of mass M follows from fact that the centripetal acceleration a_c is provided solely by the Newtonian gravity force. Thus,

$$m \times a_c = F_{\text{grav}} \quad \Longrightarrow \quad m\,\omega^2 R \;=\; \frac{GMm}{R^2} \tag{S.7}$$

and hence

$$\omega^2 \;=\; \frac{GM}{R^3}\,, \tag{S.8}$$

regardless of the satellite's own mass m. In terms of the orbital period,

$$T \;=\; \frac{2\pi}{\omega} \;=\; 2\pi\sqrt{\frac{R^3}{GM}}\,. \tag{S.9}$$

For a satellite in an elliptic orbit, the analysis is more complicated and requires calculus. However, the end result for the period is exactly as in eq. (S.9), except that the radius R should be replaced with the semi-major axis a of the elliptic orbit,

$$T \;=\; 2\pi\sqrt{\frac{a^3}{GM}}\,. \tag{S.10}$$

Now let's compare eqs. (S.10) for Triton and Luna. Denoting T_T and T_L their respective orbital periods, and a_T and a_L the semi-major axes of their orbits, we have

$$T_T \;=\; 2\pi\sqrt{\frac{a_T^3}{GM_N}}\,, \qquad T_L \;=\; 2\pi\sqrt{\frac{a_L^3}{GM_E}}\,. \tag{S.11}$$

where M_N is the Neptune's mass and M_E is the mass of the Earth. Taking the ratio of eqs. (S.11), we obtain

$$\frac{T_T}{T_L} = \sqrt{\frac{a_T^3}{GM_N}} \bigg/ \sqrt{\frac{a_L^3}{GM_E}} = \sqrt{\frac{a_T^3}{GM_N} \bigg/ \frac{a_L^3}{GM_E}} = \sqrt{\frac{a_T^3}{a_L^3} \bigg/ \frac{M_N}{M_E}}$$

and consequently

$$\left(\frac{T_T}{T_L}\right)^2 = \left(\frac{a_T}{a_L}\right)^3 \bigg/ \frac{M_N}{M_E}. \tag{S.12}$$

From this equation, we find the ratio of the two planet's masses as

$$\begin{aligned} \frac{M_N}{M_E} &= \left(\frac{a_T}{a_L}\right)^3 \bigg/ \left(\frac{T_T}{T_L}\right)^2 \\ &= \left(\frac{354,800 \text{ km}}{384,400 \text{ km}}\right)^3 \bigg/ \left(\frac{5.877 \text{ days}}{27.32 \text{ days}}\right)^2 \\ &= 17.0, \end{aligned} \tag{S.13}$$

i.e., Neptune is 17 times more massive than Earth.

PS: This is not required for this test — and the students who finish with eq. (S.13) will get full credit, — but it's good to know. The actual mass ratio is 17.2, slightly larger than in eq. (S.13), because Luna's mass does have a a small effect on its orbital motion.

While most moons have mass less than 1/1000 of the planet they orbit, the Earth–Luna system has unusually small mass ratio $M_E/M_L \approx 81$. (In the solar system, only the Pluto–Charon system has a smaller ratio.) Consequently, it becomes noticeable that Luna orbits not the Earth's center but the common center of mass of the Earth–Luna system, which is about 4700 km closer to the Luna. Hence, in eq. (S.7), the radius R in the formula for centripetal acceleration is not quite the same as the distance in the formula for the gravity force. Re-deriving the orbital equations to account for this effect, one ends up with

$$T_L = 2\pi\sqrt{\frac{a_L^3}{G(M_E + M_L)}} \tag{S.14}$$

instead of eq. (S.10). Similar corrections apply to any binary system in which the satellite's mass cannot be neglected compared to the mass of its primary.

Therefore, eq. (S.13) should be modified as

$$\frac{M_N + M_T}{M_E + M_L} = \left(\frac{a_T}{a_L}\right)^3 \Bigg/ \left(\frac{T_T}{T_L}\right)^2 = 17.0. \tag{S.15}$$

In the numerator on the left hand side, we may neglect Triton's mass M_T because it's almost 5000 times smaller than Neptune's. But Luna's mass is more noticeable compared to Earth's, so we should say that Neptune is 17 times more massive than Earth and Luna together, or about 17.2 times the mass of Earth alone.

Problem #**3**:

First solution: Assume the road has a constant uphill slope θ such that

$$\sin\theta = \frac{150\text{ m}}{2.0\text{ km}} = 0.075 \tag{S.16}$$

and that the rider pedals uphill with the constant speed. Then the net force on the rider+bike is zero, and the forward force on the bike cancels the backward component of the force of gravity,

$$f = Mg \times \sin\theta = 800\text{ N} \times 0.075 = 60\text{ N}. \tag{S.17}$$

This forward force is the static friction between the bike's tires and the ground, and it is there because the rider pushes on the pedals: The bicycle is basically a machine for transforming the rider's force on the pedals into the forward force f. It's a highly efficient machine, so the work done by the rider is approximately the same as the work done by the forward force:

$$W(f) = f \cdot \text{forward displacement } L = 60\text{ N} \cdot 2000\text{ m} = 120,000\text{ J}. \tag{S.18}$$

Alternative solution, which does not depend on the riding speed and/or the road's uphill slope being constant. For all we care, the road can be steeper in some places and less steep in others, and the rider can change his speed accordingly. Let's simply assume that the final speed at the top of the hill is the same as initial speed at the bottom, so there is no *net* change of the rider's and bike's kinetic energies, $\Delta K = 0$. Then in the absence of rolling friction, wind drag, and other resistive forces, the rider's mechanical work serves to increase the *potential* energy,

$$W(\text{rider}) \;=\; \Delta E_{\text{mech}} \;=\; \Delta U \;+\; \Delta K \;=\; \Delta U \tag{S.19}$$

where the last equality follows from assuming $\Delta K = 0$. The potential energy

$$U \;=\; Mg \cdot y \tag{S.20}$$

does not care for the horizontal motion, it depends only on the elevation y. Thus,

$$\Delta U \;=\; Mg \cdot \Delta y \;=\; 800\ \text{N} \cdot 150\ \text{m} \;=\; 120,000\ \text{J} \tag{S.21}$$

regardless of the road's length; only the net gain of elevation — the hill's height — is important.

Altogether, the mechanical work of the rider is

$$W(\text{rider}) \;=\; \Delta U \;=\; Mg \cdot \Delta y \;=\; 120,000\ \text{J}. \tag{S.22}$$

Problem #**4**:

Let's start with part (b). Consider the speed of relative motion of the two balls before and after the collision,

$$v_{\text{rel}} \;=\; |\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2| \quad \text{and} \quad v'_{\text{rel}} \;=\; \left|\vec{\mathbf{v}}'_1 - \vec{\mathbf{v}}'_2\right| \tag{S.23}$$

In an elastic collision $v'_{\text{rel}} = v_{\text{rel}}$; in a partially inelastic collision $v'_{\text{rel}} < v_{\text{rel}}$ but $v'_{\text{rel}} \neq 0$; and in a totally inelastic collision $v'_{\text{rel}} = 0$.

For the collision at hand, the relative speed before the collision was

$$v_{\text{rel}} \;=\; |(+10\ \text{m/s}) - (0\ \text{m/s})| \;=\; 10\ \text{m/s} \tag{S.24}$$

while after the collision it became

$$v'_{\text{rel}} \;=\; |(-4\ \text{m/s}) - (+4\ \text{m/s})| = \; 8\ \text{m/s}. \tag{S.25}$$

Clearly $v'_{\text{rel}} < v_{\text{rel}}$ but $v'_{\text{rel}} \neq 0$, so this collision is partially inelastic.

Now let's address part (a). In any collision, the net momentum of the two colliding bodies is conserved,

$$m_1 v_1' \;+\; m_2 v_2' \;=\; P_{\text{net}}' \;=\; P_{\text{net}} \;=\; m_1 v_1 \;+\; m_2 v_2 \,. \tag{S.26}$$

For the collision in question we know both ball's velocities both before and after the collision, but we do not know the mass m_2 of the brass ball. To find it, we rewrite eq. (S.26) as

$$m_2 \times (v_2' - v_2) \;=\; m_1 \times (v_1 - v_1'), \tag{S.27}$$

which gives us the *ratio* of the two masses:

$$\frac{m_2}{m_1} \;=\; \frac{v_1 - v_1'}{v_2' - v_2} \;=\; \frac{(+10\text{ m/s}) - (-4\text{ m/s})}{(+4\text{ m/s}) - (0\text{ m/s})} \;=\; 3.5\,. \tag{S.28}$$

Hence, given the steel ball's mass $m_1 = 120$ g, the brass ball's mass is $m_2 = 3.5\times m_1 = 420$ g.

Alternative solution for part (b):

In an elastic collision, the net kinetic energy after the collision is the same as before the collision, but in an inelastic collision some kinetic energy is lost (*i.e.*, becomes heat). Now that we know the masses of both balls, we can check what happens to the kinetic energy in the collision of interest.

Before the collision, the net kinetic energy of the two balls was

$$\begin{aligned} K &= \tfrac{1}{2}m_1 v_1^2 \;+\; \tfrac{1}{2}m_2 v_2^2 \\ &= \tfrac{1}{2}(120\text{ g})(10.0\text{ m/s})^2 \;+\; 0 \\ &= 6.0\text{ J}. \end{aligned} \tag{S.29}$$

After the collision, the net kinetic energy became

$$\begin{aligned} K' &= \tfrac{1}{2}m_1 v_1'^2 \;+\; \tfrac{1}{2}m_2 v_2'^2 \\ &= \tfrac{1}{2}(120\text{ g})(-4.0\text{ m/s})^2 \;+\; \tfrac{1}{2}(420\text{ g})(+4.0\text{ m/s})^2 \\ &= 4.32\text{ J}. \end{aligned} \tag{S.30}$$

We see that some energy is lost in this collision, so it is inelastic rather than elastic.

We should also check that this collision is not totally inelastic, but this is evident from the fact that the two balls do not stick together but move away in opposite directions.

Problem #**5**:

First, let's relate $v^{\rm bucket}$ to $\omega^{\rm axle}$. When the axle rotates through some angle $\Delta\phi$, every link of the chain wrapped around it moves through distance $L = R \times \Delta\phi$. The un-wrapped part of the chain moves through the same distance L but in the downward direction, and consequently the bucket moves down by

$$\Delta y^{\rm bucket} \;=\; -L \;=\; -R \times \Delta\phi^{\rm axle}. \tag{S.31}$$

Dividing both sides of this equation by the times interval Δt we get

$$\frac{\Delta y^{\rm bucket}}{\Delta t} \;=\; -R \times \frac{\Delta\phi^{\rm axle}}{\Delta t}\,, \tag{S.32}$$

and in the limit of a very short time interval $\Delta t \to 0$ this becomes

$$v_y^{\rm bucket} \;=\; -R \times \omega^{\rm axle}. \tag{S.33}$$

Now let's compare the kinetic energies of the bucket going down and of the rotating axle. The bucket has

$$K^{\rm bucket} \;=\; \frac{mv^2}{2} \;=\; mR^2\omega^2 \tag{S.34}$$

where the second equality comes from eq. (S.33), while the axle has

$$K^{\rm axle} \;=\; \frac{I\omega^2}{2} \;=\; \frac{MR^2\omega^2}{4} \tag{S.35}$$

because a cylindrical axle has $I = \frac{1}{2}MR^2$. Altogether,

$$K^{\rm net} \;=\; K^{\rm bucket} \;+\; K^{\rm axle} \;=\; \left(\frac{mR^2}{2} \;+\; \frac{MR^2}{4}\right) \times \omega^2. \tag{S.36}$$

Finally, consider the net mechanical energy of the bucket and the axle,

$$E \;=\; K^{\rm bucket} \;+\; K^{\rm axle} \;+\; U^{\rm bucket} \;=\; \left(\frac{mR^2}{2} \;+\; \frac{MR^2}{4}\right) \times \omega^2 \;+\; mgy^{\rm bucket} \tag{S.37}$$

(I don't include the potential energy of the axle here because it does not change — the axle does not move up or down.) In the absence of friction, the net mechanical energy is conserved, hence

$$\left(\frac{mR^2}{2} + \frac{MR^2}{4}\right) \times \omega^2 + mgy^{\rm bucket} = {\rm const.} \tag{S.38}$$

When the bucket goes down, its potential energy decreases while the kinetic energies of the bucket and the axle increase. Since initially $v_0^{\rm bucket} = 0$ and hence $\omega_0^{\rm axle} = 0$, the system starts with zero kinetic energy. As the bucket goes down, the *net* kinetic energy becomes

$$K^{\rm net} = -\Delta U^{\rm bucket} = -mg\Delta y^{\rm bucket} = -12\,{\rm kg} \times 9.8\,{\rm m/s}^2 \times (-4.0\,{\rm m}) = +\,470\,{\rm J}. \tag{S.39}$$

In light of eq. (S.36), this gives us the axle's angular velocity as

$$\omega = \sqrt{\frac{K}{\frac{1}{2}mR^2 + \frac{1}{4}MR^2}} = \sqrt{\frac{470\,{\rm J}}{\big(\frac{1}{2}(12\,{\rm kg}) + \frac{1}{4}(24\,{\rm kg})\big) \times (0.11\,{\rm m})^2}} = 57\,{\rm rad/s}, \tag{S.40}$$

which corresponds to rotation rate $\frac{\omega}{2\pi} = 9$ revolutions per second.

The linear velocity of the bucket follows via eq. (S.33):

$$v_y^{\rm bucket} = -R \times \omega^{\rm axle} = 0.11\,{\rm m} \times 67\,{\rm s}^{-1} = 6.3\,{\rm m/s} \approx 14\,{\rm MPH}. \tag{S.41}$$

PS: You don't have to do this for the exam, but it's good to know. Analytically,

$$\left|v_y^{\rm bucket}\right| = R \times \omega^{\rm axle} = R \times \sqrt{\frac{K = mg|\Delta y|}{\frac{1}{2}mR^2 + \frac{1}{4}MR^2}} = \frac{\sqrt{2g|\Delta y|}}{\sqrt{1 + \frac{M}{2m}}}. \tag{S.42}$$

When the axle is much lighter than the bucket full of water, $M \ll m$, it does little to slow down the bucket's fall. Accordingly, on the RHS of eq. (S.42) the denominator becomes approximately 1 and hence

$$\text{for } M \ll m, \quad |v| \approx \sqrt{2g|\Delta y|} = v(\text{free fall from height } |\Delta y|). \tag{S.43}$$

But when the bucket is much heavier than the bucket, $M \gg m$, the bucket gains much smaller velocity

$$|v^{\rm bucket}| = \frac{v^{\rm free\,fall}}{\sqrt{1 + \frac{M}{2m}}} \ll v^{\rm free\,fall} \tag{S.44}$$

because most of the potential energy of the bucket goes to the kinetic energy of the axle rather than the bucket.