Problem **1**:

Out in space, there are no forces acting on the balloon except gravity and hence no torques (with respect to an axis through the center of mass). Consequently, the *angular momentum* of the balloon is conserved,

$$L = I \times \omega = \text{const.} \tag{S.1}$$

The angular momentum of the balloon depends on its radius; approximating the balloon as a thin spherical shell of uniform density and thickness, we have

$$I = \tfrac{2}{3} M R^2. \tag{S.2}$$

When the balloon shrinks (because the air inside it cools down), the moment of inertia decreases,

$$I = \tfrac{2}{3} M \times R^2 \textit{ to } I' = \tfrac{2}{3} M' \times R'^2; \tag{S.3}$$

specifically,

$$\frac{I'}{I} = \left(\frac{R'}{R}\right)^2 = \left(\frac{1\text{ ft}}{2\text{ ft}}\right)^2 = \frac{1}{4}\,. \tag{S.4}$$

But the angular momentum of the balloon is conserved,

$$L = I \times \omega = I' \times \omega'. \tag{S.5}$$

Thus, when the moment of inertia decreases, the angular velocity must increase to keep the angular momentum constant; specifically

$$\frac{\omega'}{\omega} = \frac{I}{I'} = 4. \tag{S.6}$$

Consequently, the rotational period of the balloon — the time

$$T = \frac{2\pi}{\omega} \tag{S.7}$$

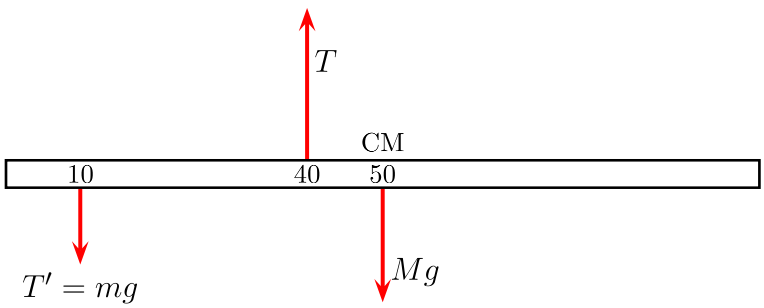
it takes to make one complete turn around its spin axis — decreases as

$$\frac{T'}{T} = \frac{\omega}{\omega'} = \frac{1}{4}. \tag{S.8}$$

Thus, if the balloon had originally rotated once every $T = 40$ s, then after it has cooled down, it was rotating 4 times faster, once every $T' = T/4 = 10$ s.

Problem **2**:

There are three forces acting on the meter-stick: the tension T of the upper string, the tension $T' = mg$ (where $m = 50$ g) of the lower string, and the meter-stick's own weight Mg. Here is the force diagram:



Although the weight force Mg is distributed all over the meter-stick, for the purpose of calculating torques, we treat it as acting at the center of mass, and that's what the diagram shows. By symmetry, the center of mass is in the middle of the meter-stick, at the 50 cm mark.

The meter-stick is in equilibrium, so the net force and the net torque on it must be zero,

$$\sum F = 0, \qquad \sum \tau = 0. \tag{S.9}$$

In the torque condition, we may calculate the torques relative to any pivot point we like (as long as it's the same point for all the forces), so let's consider the net torque relative to the 40 cm mark where the upper string is attached. With this choice, the tension T of the upper string has zero lever arm, the tension $T' = mg$ of the lower string has lever arm

40 cm − 10 cm = 30 cm in the counterclockwise direction, and the meter-stick's own weight Mg has lever arm 50 cm − 40 cm = 10 cm in the clockwise direction. Consequently, the net torque is

$$\tau^{\rm net} \equiv \tau(T) + \tau(mg) + \tau(Mg) = 0 - mg\times 30{\rm cm} + Mg\times 10{\rm cm}. \tag{S.10}$$

Demanding that this net torque vanishes, we obtain

$$Mg\times 10{\rm cm} - mg\times 30{\rm cm} = 0, \tag{S.11}$$

and consequently the meter-stick's mass is

$$M = m\times\frac{30\ {\rm cm}}{10\ {\rm cm}} = m\times 3 = 50\ {\rm g}\times 3 = 150\ {\rm g}. \tag{S.12}$$

Problem **3**:

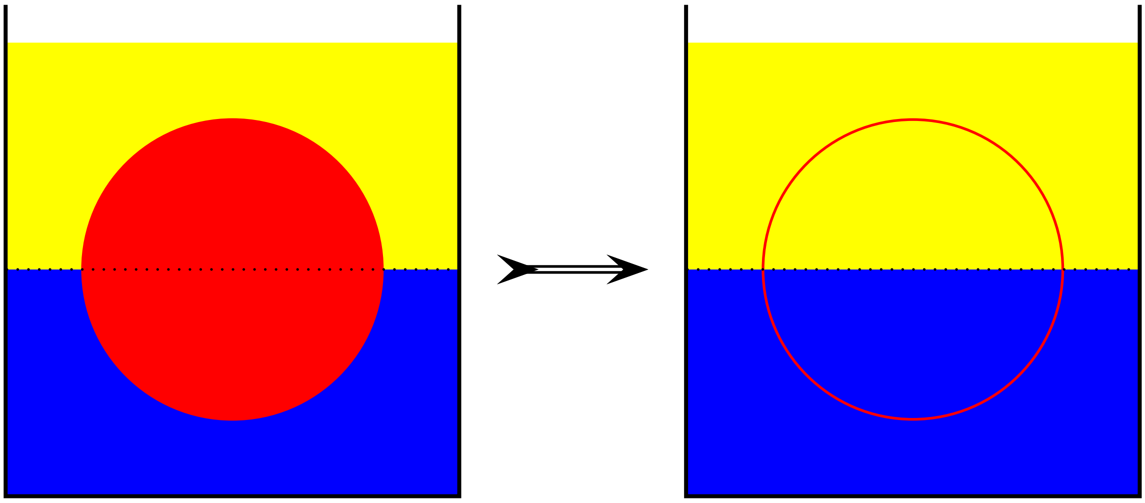
The plastic ball is in equilibrium because its weight mg is opposed by the equal buoyant force; the real problem is figuring out the buoyant force on a body floating at the interface of two liquids, oil and water.

Let's go back to the origin of the Archimedes Law. When a body is immersed in a liquid, the buoyant force on the body comes from the net force of liquid pressure from all sides of the body. Those pressure forces act on the surface of the body and don't care what's inside it, plastic, metal, air, or more liquid. Suppose there was nothing but liquid inside that surface, same as the liquid outside it. Then this liquid would be in equilibrium, so the net buoyant force must precisely cancel its weight, hence

$$F_B = V\rho_{\rm liquid}g. \tag{S.13}$$

And any other body occupying the same volume instead of the liquid would feel exactly the same buoyant force.

Now consider the plastic ball floating at the interface of oil and water. The buoyant force on the ball comes from pressure forces of oil and water, and those forces don't care what's inside this surface, plastic or more oil and water. Suppose instead of plastic, we fill that spherical boundary with more oil and water, oil in the top half of the ball and water in the bottom half, so the oil/water boundary inside the sphere would be level with the boundary outside the sphere.



Clearly, in this situation both the oil and the water would be in equilibrium. Therefore, the net buoyant force on the spherical boundary (the red circle on the right picture) would be equal to the net weight of oil and water inside that sphere,

$$F_B^{\text{net}} = g\rho_{\text{oil}} \times \frac{V_{\text{ball}}}{2} + g\rho_{\text{water}} \times \frac{V_{\text{ball}}}{2}. \tag{S.14}$$

And when we go back to the left picture and replace the oil and water inside the sphere with the solid plastic, the buoyant force stays exactly the same.

If the plastic ball stays in equilibrium in the half-in-oil, half-in-water position, then the buoyant force (S.14) on the ball is equal to the ball's own weight,

$$F_B^{\text{net}} = gm = g\rho_{\text{plastic}} \times V_{\text{ball}}. \tag{S.15}$$

In light of eq. (S.14), this gives us

$$g\rho_{\text{oil}} \times \frac{V_{\text{ball}}}{2} + g\rho_{\text{water}} \times \frac{V_{\text{ball}}}{2} = g\rho_{\text{plastic}} \times V_{\text{ball}}. \tag{S.16}$$

Dividing both sides of this equation by gV_{ball}, we get

$$\rho_{\text{plastic}} = \frac{\rho_{\text{oil}}}{2} + \frac{\rho_{\text{water}}}{2} = \frac{0.80\ \text{g/cm}^3}{2} + \frac{1.00\ \text{g}}{2} = 0.90\ \text{g/cm}^3. \tag{S.17}$$

<u>PS</u>: More generally, for a body floating at the interface of two liquids — but with more volume immersed in one liquid that the other — the buoyant force is

$$F_B^{\text{net}} = g\rho_{\text{liquid}\#1} \times V_{\text{in}\,1} + g\rho_{\text{liquid}\#2} \times V_{\text{in}\,2}\,. \tag{S.18}$$

If that body floats in equilibrium, than its average density is

$$\bar{\rho}_{\text{body}} \equiv \frac{M}{V_{\text{total}}} = \rho_{\text{liquid}\#1} \times \frac{V_{\text{in}\,1}}{V_{\text{total}}} + \rho_{\text{liquid}\#2} \times \frac{V_{\text{in}\,2}}{V_{\text{total}}}\,. \tag{S.19}$$

For example, if 3/4 of the ball's volume were immersed in water and the remaining 1/4 of the volume were immersed in oil, then we would have $\rho_{\text{plastic}} = (3/4)\rho_{\text{water}} + (1/4)\rho_{\text{oil}} = 0.95\ \text{g/cm}^3$.

<u>Problem **4**</u>:

Consider the two ends of the hot water pipe: its beginning inside the boiler (1), and its end at the faucet's opening (2). In the absence of viscosity, the pressures at the two ends are related by the Bernoulli equation,

$$P_1 + \tfrac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \tfrac{1}{2}\rho v_2^2 + \rho g y_2\,. \tag{S.20}$$

Since the faucet is open to the air, $P_2 = P_{\text{atm}}$. Consequently, the pressure difference

$$P_1 - P_2 = P_1 - P_{\text{atm}} = P_1^{\text{gauge}} \tag{S.21}$$

is equal to the *gauge pressure* of water in the boiler. In light of the Bernoulli equation (S.20), this gives us

$$P_1^{\text{gauge}} = P_1 - P_2 = \tfrac{1}{2}\rho\big(v_2^2 - v_1^2\big) + \rho g(y_2 - y_1). \tag{S.22}$$

We know that the basement is 5.0 m below the faucet, thus $y_2 - y_1 = +5.0$ m. But we don't know the velocities of water in the faucet or in the boiler. To find them out, we can use the continuity equation,

$$\mathcal{F} \;=\; A_1 \times v_1 \;=\; A_2 \times v_2 \tag{S.23}$$

where $\mathcal{F}$ is the flow rate through the pipe, A_2 is the cross-sectional area of the faucet, and A_1 is the cross-sectional area of the water inside the boiler. The problem gives us $A_2 = 1.0\ \mathrm{cm}^2$ and $\mathcal{F} = 1.0$ L/s, so we can find the velocity of water in the faucet as

$$v_2 \;=\; \frac{\mathcal{F}}{A_2} \;=\; \frac{1.0\ \mathrm{L/s} \;=\; 1000\ \mathrm{cm}^3/\mathrm{s}}{1.0\ \mathrm{cm}^2} \;=\; 1000\ \mathrm{cm/s} \;=\; 10\ \mathrm{m/s}. \tag{S.24}$$

We don't know the cross-sectional area of the boiler, but obviously it's much larger than the faucet's cross-section, $A_1 \gg A_2$, hence

$$v_1 \;=\; \frac{\mathcal{F}}{A_1} \;\ll\; \frac{\mathcal{F}}{A_2} \;=\; v_2 \tag{S.25}$$

and consequently

$$\tfrac{1}{2}\rho\big(v_2^2 - v_1^2\big) \;\approx\; \tfrac{1}{2}\rho v_2^2\,. \tag{S.26}$$

Using this approximation in eq. (S.22), we get

$$\begin{aligned} P_1^{\mathrm{gauge}} \;&=\; \tfrac{1}{2}\rho\big(v_2^2 - v_1^2\big) \;+\; \rho g(y_2 - y_1) \\ &\approx\; \tfrac{1}{2}\rho v_2^2 \;+\; \rho g(y_2 - y_1) \\ &=\; \tfrac{1}{2}(1000\ \mathrm{kg/m}^3)(10\ \mathrm{m/s})^2 \;+\; (1000\ \mathrm{kg/m}^3)(9.8\ \mathrm{m/s}^2)(5.0\ \mathrm{m}) \\ &=\; 99,000\ \mathrm{Pa} \;=\; 0.99\ \mathrm{bar} \;=\; 0.98\ \mathrm{atm}. \end{aligned} \tag{S.27}$$

Problem **5**:

(a) Using the universal gas equation

$$PV \;=\; nRT, \tag{S.28}$$

we can find the temperature of the gas from its volume and pressure as

$$T \;=\; \frac{PV}{nR} \;=\; \frac{(3.00\ \cdot 10^{-3}\ \mathrm{m}^3)(8.00\cdot 10^5\ \mathrm{Pa})}{(1\ \mathrm{mol})(8.314\ \mathrm{J/K/mol})} \;=\; 288.7\ \mathrm{K}. \tag{S.29}$$

This is the *absolute temperature* of the helium gas; it translates to the everyday degrees as 15.5°C or 60°F.

(b) The average kinetic energy of a gas molecule — or in our case a helium atom — is related to the gas's absolute temperature as

$$\left\langle \tfrac{1}{2}mv^2 \right\rangle_{\text{avg}} \;=\; \tfrac{3}{2}kT \tag{S.30}$$

where $k = 1.38 \cdot 10^{-23}\ J/K$ is the Boltzmann's constant. For the helium tank in question,

$$\left\langle \tfrac{1}{2}mv^2 \right\rangle_{\text{avg}} \;=\; \tfrac{3}{2}(1.38 \cdot 10^{-23}\ \text{J/K})(288.7\ \text{K}) \;=\; 5.98 \cdot 10^{-21}\ \text{J}. \tag{S.31}$$

One mol of helium has $N = N_A = 6.022\ \cdot 10^{23}$ atoms, so their net kinetic energy is

$$K_{\text{net}} \;=\; N_A \times \left\langle K_1 = \tfrac{1}{2}mv^2 \right\rangle_{\text{avg}} \;=\; (6.022\ \cdot 10^{23}) \times (5.98 \cdot 10^{-21}\ \text{J}) \;=\; 3600\ \text{J}. \tag{S.32}$$

<u>Alternative solution</u>:

Actually, given the volume and the pressure of a gas, we can find the net kinetic energy of (the linear motion of) all the molecules in the gas without knowing the temperature or the molecular / atomic weight of the gas, or even its amount. Most generally,

$$\left\langle \tfrac{1}{2}mv^2 \right\rangle_{\text{avg}} \;=\; \tfrac{3}{2}kT \tag{S.33}$$

and hence

$$K_{\text{net}} \;=\; N \times \left\langle \tfrac{1}{2}mv^2 \right\rangle_{\text{avg}} \;=\; N \times \tfrac{3}{2}kT \tag{S.34}$$

where N is the net number of gas molecules, whatever it happens to be. By the universal gas law,

$$PV \;=\; nRT \;=\; NkT \tag{S.35}$$

(note that $N = n \times N_A$ while $R = N_A \times k$), hence

$$K_{\text{net}} \;=\; \tfrac{3}{2}NkT \;=\; \tfrac{3}{2}PV. \tag{S.36}$$

For the helium gas in question,

$$K_{\text{net}} \;=\; \tfrac{3}{2}(3.00\ \cdot 10^{-3}\ \text{m}^3)(8.00 \cdot 10^5\ \text{Pa}) \;=\; 3600\ \text{Pa} \cdot \text{m}^3 \;=\; 3600\ \text{J}. \tag{S.37}$$