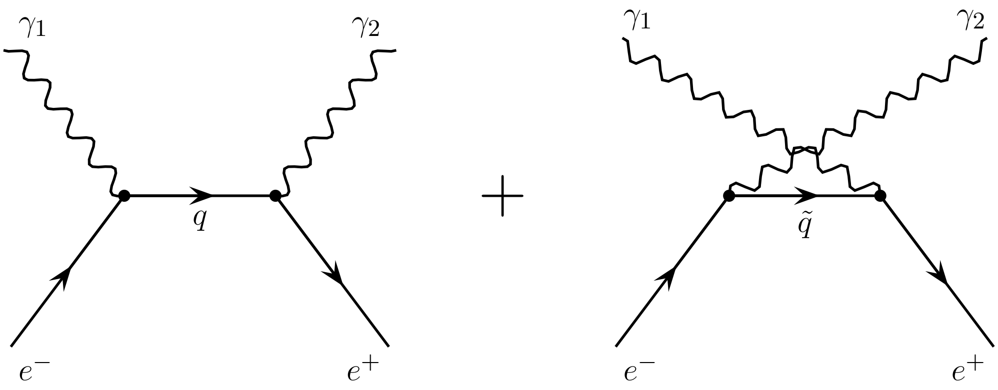
ANNIHILATION

In these notes I explain the $e^+e^- \to \gamma\gamma$ annihilation process. At the tree level of QED, there are two diagrams related by interchanging of the two photons in the final state:



(1)

The net amplitude due to these diagrams is

$$
\begin{aligned}
\mathcal{M} &= e^*_\mu(k_1,\lambda_1)\, e^*_\nu(k_2,\lambda_2) \times \mathcal{M}^{\mu\nu}, \\
\mathcal{M}^{\mu\nu} &= \mathcal{M}_1^{\mu\nu} \;+\; \mathcal{M}_2^{\mu\nu}, \\
i\mathcal{M}_1^{\mu\nu} &= \bar{v}(e^+)(ie\gamma^\nu)\frac{i}{\not{q}-m}(ie\gamma^\mu)u(e^-), \\
i\mathcal{M}_2^{\mu\nu} &= \bar{v}(e^+)(ie\gamma^\mu)\frac{i}{\not{\tilde{q}}-m}(ie\gamma^\nu)u(e^-),
\end{aligned}
\tag{2}
$$

where $q = p_- - k_1 = k_2 - p_+$ and $\tilde{q} = p_- - k_2 = k_1 - p_+$. Note the opposite orders of the γ^μ and γ^ν vertices in the $\mathcal{M}_1$ and the $\mathcal{M}_2$ amplitudes. We may re-write these amplitudes without matrix denominators using

$$
\frac{1}{\not{q}-m} = \frac{\not{q}+m}{q^2-m^2} = \frac{\not{q}+m}{t-m^2} \quad\text{and}\quad \frac{1}{\not{\tilde{q}}-m} = \frac{\not{\tilde{q}}+m}{q^2-m^2} = \frac{\not{\tilde{q}}+m}{u-m^2}\,. \tag{3}
$$

Consequently,

$$
\begin{aligned}
\mathcal{M}_1^{\mu\nu} &= \frac{-e^2}{t-m^2} \times \bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u, \\
\mathcal{M}_2^{\mu\nu} &= \frac{-e^2}{u-m^2} \times \bar{v}\gamma^\mu(\not{\tilde{q}}+m)\gamma^\nu u.
\end{aligned}
\tag{4}
$$

Ward Identity

Before we go any further, lets check the Ward identities for the annihilation amplitude: for the first photon we should have $k_{1\mu}\mathcal{M}^{\mu\nu} = 0$, and for the second photon $k_{2\nu}\mathcal{M}^{\mu\nu} = 0$. Let's start with the first photon and the first diagram. Multiplying the second factor in the first eq. (4) by $k_{1\mu}$, we have

$$\begin{aligned}
\bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u \times k_{1\mu} &= \bar{v}\gamma^\nu(\not{p}_- - \not{k}_1 + m)\not{k}_1 u \\
&= \bar{v}\gamma^\nu(\not{p}_- + m)\not{k}_1 u \quad \langle\!\langle\text{because } \not{k}_1\not{k}_1 = k_1^2 = 0\rangle\!\rangle \\
&= \bar{v}\gamma^\nu\Big(2(p_-k_1) - \not{k}_1(\not{p}_- - m)\Big)u \\
&= 2(p_-k_1)\times\bar{v}\gamma^\nu u \quad \langle\!\langle\text{because } (\not{p}_- - m)u = 0\rangle\!\rangle \\
&= (m^2 - t)\times\bar{v}\gamma^\nu u
\end{aligned} \tag{5}$$

and consequently

$$\mathcal{M}_1^{\mu\nu}\times k_{1\mu} = +e^2\times\bar{v}\gamma^\nu u\,. \tag{6}$$

Note the non-zero right hand side — the first diagram does not satisfy the Ward identity all by itself. As for the second diagram, we have

$$\begin{aligned}
\bar{v}\gamma^\mu(\tilde{\not{q}}+m)\gamma^\nu u \times k_{1\mu} &= \bar{v}\,\not{k}_1(\not{k}_1 - \not{p}_+ + m)\gamma^\nu u \\
&= \bar{v}\,\not{k}_1(-\not{p}_+ + m)\gamma^\nu u \quad \langle\!\langle\text{because } \not{k}_1\not{k}_1 = k_1^2 = 0\rangle\!\rangle \\
&= \bar{v}\Big(-2(p_+k_1) + (\not{p}_+ + m)\not{k}_1\Big)\gamma^\nu u \\
&= -2(p_+k_1)\times\bar{v}\gamma^\nu u \quad \langle\!\langle\text{because } \bar{v}(\not{p}_+ + m) = 0\rangle\!\rangle \\
&= -(m^2 - u)\times\bar{v}\gamma^\nu u
\end{aligned} \tag{7}$$

and consequently

$$\mathcal{M}_2^{\mu\nu}\times k_{1\mu} = -e^2\times\bar{v}\gamma^\nu u\,. \tag{8}$$

Again we have a non-zero result — the second diagram also does not satisfy the Ward identity all by itself. However, the right hand sides of eqs. (6) and (8) cancel each other, so *together*, the two diagrams do satisfy the Ward identity:

$$\mathcal{M}^{\mu\nu} \times k_{1\mu} \;=\; \mathcal{M}_1^{\mu\nu} \times k_{1\mu} \;+\; \mathcal{M}_2^{\mu\nu} \times k_{1\mu} \;=\; 0. \tag{9}$$

This is an example of a general rule: The Ward Identity does not work diagram by diagram but only for entire amplitudes, or for partial sums of all diagrams related by permutations of photonic vertices on the same fermionic line.

The Ward identity $\mathcal{M}^{\mu\nu} \times k_{2\nu} = 0$ for the second photon works similarly to the first, and I see no point in repeating the argument. Indeed, it would be an exactly similar argument because the net annihilation amplitude is symmetric with respect to the two photons.

Summing over the Spins and Polarizations

Earlier in class I explained how to use Ward identities to sum $|\mathcal{M}|^2$ over polarizations of the two photons:

$$\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \;=\; +\mathcal{M}^{\mu\nu}\mathcal{M}^*_{\mu\nu}\,. \tag{10}$$

Combining the two diagrams, we have

$$\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \;=\; +\mathcal{M}^{\mu\nu}\mathcal{M}^*_{\mu\nu} \;=\; \mathcal{M}_1^{\mu\nu}\mathcal{M}^*_{1\mu\nu} \;+\; \mathcal{M}_2^{\mu\nu}\mathcal{M}^*_{2\mu\nu} \;+\; 2\,\mathrm{Re}\,\mathcal{M}_1^{\mu\nu}\mathcal{M}^*_{2\mu\nu}\,. \tag{11}$$

Note that this formula works despite the fact that $\mathcal{M}_1^{\mu\nu}$ and $\mathcal{M}_2^{\mu\nu}$ do not satisfy the Ward identities by themselves — it's enough that the sum $\mathcal{M}_1^{\mu\nu} + \mathcal{M}_2^{\mu\nu}$ satisfies the identities. Thus, in light of eqs. (4),

$$\begin{aligned}\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \;=\;& \frac{e^4}{(t-m^2)^2} \times \bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u \times \bar{u}\gamma_\mu(\not{q}+m)\gamma_\nu v \\ &+\; \frac{e^4}{(u-m^2)^2} \times \bar{v}\gamma^\mu(\tilde{\not{q}}+m)\gamma^\nu u \times \bar{u}\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu v \\ &+\; \frac{2e^4}{(t-m^2)(u-m^2)} \times \mathrm{Re}\Big(\bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u \times \bar{u}\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu v\Big).\end{aligned} \tag{12}$$

This takes care of the photon polarizations. The next step is to average over spins of the initial electron and positron. Proceeding is usual, we have

$$\begin{aligned}\overline{|\mathcal{M}|^2} &\equiv \tfrac{1}{4}\sum_{s_-,s_+}\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2 \\ &= \frac{e^4}{(t-m^2)^2}\times A_{11} \;+\; \frac{e^4}{(u-m^2)^2}\times A_{22} \;+\; \frac{2e^4}{(t-m^2)(u-m^2)}\times \mathrm{Re}\,A_{12}\,,\end{aligned} \tag{13}$$

where

$$\begin{aligned} A_{11} &= \tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m)\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_- + m)\gamma_\mu(\not{q}+m)\gamma_\nu\Big), \\ A_{22} &= \tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m)\gamma^\mu(\tilde{\not{q}}+m)\gamma^\nu(\not{p}_- + m)\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu\Big), \\ A_{12} &= \tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m)\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_- + m)\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu\Big), \end{aligned} \tag{14}$$

Traceology 1

Our next task is to evaluate the traces (14). Let's start with the A_{11}.

Back in homework set #5, you saw that $\gamma^\mu\gamma_\mu = 4$ and $\gamma^\mu\not{p}\gamma_\mu = -2\not{p}$. Applying these formulae to the expression inside the trace in A_{11}, we have

$$\gamma^\mu(\not{p}_- + m)\gamma_\mu \;=\; -2(\not{p}_- - 2m), \qquad \gamma_\nu(\not{p}_+ - m)\gamma^\nu \;=\; -2(\not{p}_+ + 2m), \tag{15}$$

and consequently

$$A_{11} \;=\; \mathrm{Tr}\Big((\not{p}_+ + 2m)(\not{q}+m)(\not{p}_- - 2m)(\not{q}+m)\Big). \tag{16}$$

Next, we expand the parentheses inside this trace and throw away terms with odd numbers of momenta $\not{p}$ or $\not{q}$. This gives us

$$\begin{aligned} A_{11} &= \mathrm{Tr}(\not{p}_+\not{q}\not{p}_-\not{q}) \;+\; m^2\,\mathrm{Tr}(\not{p}_+\not{p}_-) \;-\; 4m^2\,\mathrm{Tr}(\not{q}\not{q}) \\ &\quad +\; 2\times 2m^2\,\mathrm{Tr}(\not{p}_-\not{q}) \;-2\times 2m^2\,\mathrm{Tr}(\not{p}_+\not{q}) \;-\; 4m^4\,\mathrm{Tr}(1) \\ &= 8(p_+q)\times(p_-q) \;-\; 4(p_+p_-)\times q^2 \;+\; 4m^2(p_+p_-) \\ &\quad -\; 16m^2q^2 \;+\; 16m^2(p_-q) \;-\; 16m^2(p_+q) \;-\; 16m^4\,. \\ &= 8(p_+q)(p_-q) \;-\; 4(q^2-m^2)\times(p_+p_-) \;-\; 16m^2\times\big(q^2 \;-\; (p_-q) \;+\; (p_+q) \;+\; m^2\big)\,. \end{aligned} \tag{17}$$

We may further simplify this formula by expressing all the momenta products in terms of the Mandelstam's variables s, t, and u. Using $p_-^2 = p_+^2 = m^2$ and $k_1^2 = k_2^2 = 0$, we have

$$\begin{aligned} q^2 &= (p_- - k_1)^2 = t, \\ qp_- &= (p_- - k_1)p_- = m^2 - p_-k_1 = m^2 + \tfrac{1}{2}(t - m^2) = +\tfrac{1}{2}(m^2 + t), \\ qp_+ &= (k_2 - p_+)p_+ = p_+k_2 - m^2 = -\tfrac{1}{2}(t - m^2) - m^2 = -\tfrac{1}{2}(t + m^2), \\ p_-p_+ &= \tfrac{1}{2}(s - 2m^2). \end{aligned} \tag{18}$$

Consequently, on the last line of eq. (17), the last term vanishes —

$$q^2 - (p_-q) + (p_+q) + m^2 = t - \tfrac{1}{2}(t + m^2) - \tfrac{1}{2}(t + m^2) + m^2 = 0 \tag{19}$$

— while the remaining terms add up to

$$\begin{aligned} A_{11} &= 8(p_+q)(p_-q) - 4(q^2 - m^2) \times (p_+p_-) \\ &= -2(t + m^2)^2 - 2(t - m^2) \times (s - 2m^2 = -t - u) \\ &= -2t^2 - 4t\,m^2 - 2m^4 + 2t^2 + 2tu - 2t\,m^2 - 2u\,m^2 \\ &= 2tu - 6tm^2 - 2um^2 - 2m^4 \\ &= 2(t - m^2)\,(u - 3m^2) - 8m^4. \end{aligned} \tag{20}$$

This completes our evaluation of the first trace.

Now consider the second trace A_{22}. Instead of working through the calculation, we may use the photon exchange / crossing symmetry between the two diagrams (1). This symmetry exchanges $t \leftrightarrow u$ and also $A_{11} \leftrightarrow A_{22}$, thus

$$A_{22} = 2(u - m^2)\,(t - 3m^2) - 8m^4. \tag{21}$$

Traceology 2

Now we need to evaluate the third trace A_{12} which accounts for the interference between the two diagrams (1). This trace is more complicated, so let's start by simplifying the

$\gamma^\nu \cdots \gamma_\nu$ part. Back in homework #5, we had

$$\gamma^\nu \not{a} \gamma_\nu \;=\; -2\not{a}, \qquad \gamma^\nu \not{a}\not{b} \gamma_\nu \;=\; 4(ab), \qquad \gamma^\nu \not{a}\not{b}\not{c} \gamma_\nu \;=\; -2\not{c}\not{b}\not{a}, \tag{22}$$

which now gives us

$$\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_- + m)\gamma_\nu \;=\; -2m^2\gamma^\mu \;+\; 4m(q+p_-)^\mu \;-\; 2\not{p}_-\gamma^\mu\not{q}. \tag{23}$$

Plugging this formula into eq. (14) for the A_{12} and remembering that we need an even number of slash-momentum or γ^μ factors inside the trace, we obtain

$$\begin{aligned}
A_{12} \;&=\; \tfrac{1}{4}\,\mathrm{Tr}\Big(\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_- + m)\gamma_\nu \times (\tilde{\not{q}}+m)\gamma_\mu(\not{p}_+ - m)\Big)\\
&=\; \mathrm{Tr}\left(\Big[m(q+p_-)^\mu \;-\; \tfrac{1}{2}\big(m^2\gamma^\mu + \not{p}_-\gamma^\mu\not{q}\big)\Big] \times \Big[m\big(\gamma_\mu\not{p}_+ - \tilde{\not{q}}\gamma_\mu\big) + \big(\tilde{\not{q}}\gamma_\mu\not{p}_+ \;-\; m^2\gamma_\mu\big)\Big]\right)\\
&=\; m^2(q+p_-)^\mu \times \mathrm{Tr}\big(\gamma_\mu\not{p}_+ - \tilde{\not{q}}\gamma_\mu\big) \;-\; \tfrac{1}{2}\,\mathrm{Tr}\Big(\big(m^2\gamma^\mu + \not{p}_-\gamma^\mu\not{q}\big) \times \big(\tilde{\not{q}}\gamma_\mu\not{p}_+ \;-\; m^2\gamma_\mu\big)\Big)\\
&=\; m(q+p_-)^\mu \times 4m(p_+ - \tilde{q})_\mu\\
&\quad\;-\; \tfrac{1}{2}\,\mathrm{Tr}\Big(\not{p}_-\gamma^\mu\not{q}\tilde{\not{q}}\gamma_\mu\not{p}_+ \;-\; m^2\not{p}_-\gamma^\mu\not{q}\gamma_\mu \;+\; m^2\gamma^\mu\tilde{\not{q}}\gamma_\mu\not{p}_+ \;-\; m^4\gamma^\mu\gamma_\mu\Big)\\
&=\; 4m^2(q+p_-)^\mu(p_+ - \tilde{q})_\mu\\
&\quad\;-\; \tfrac{1}{2}\,\mathrm{Tr}\Big(4(q\tilde{q})\not{p}_-\not{p}_+ \;+\; 2m^2\not{p}_-\not{q} \;-\; 2m^2\tilde{\not{q}}\not{p}_+ \;-\; 4m^4\Big)\\
&=\; 4m^2\Big(-(q\tilde{q}) \;+\; (qp_+) \;-\; (\tilde{q}p_-) \;+\; (p_-p_+)\Big)\\
&\quad\;-\; 8(q\tilde{q})(p_-p_+) \;-\; 4m^2(p_-q) \;+\; 4m^2(\tilde{q}p_+) \;+\; 8m^4.
\end{aligned} \tag{24}$$

To simplify this rather messy formula, we need to work out the kinematics. Besides eqs. (18), we have

$$\begin{aligned}
\tilde{q}p_- \;&=\; (p_- - k_2)p_- \;=\; m^2 \;-\; k_2p_- \;=\; m^2 \;+\; \tfrac{1}{2}(u-m^2) \;=\; +\tfrac{1}{2}(u+m^2),\\
\tilde{q}p_+ \;&=\; (k_1 - p_+)p_+ \;=\; k_1p_+ \;-\; m^2 \;=\; -\tfrac{1}{2}(u-m^2) \;-\; m^2 \;=\; -\tfrac{1}{2}(u+m^2),\\
\tilde{q}q \;&=\; (p_- - k_2)(p_- - k_1) \;=\; p_-^2 \;-\; p_-(k_1+k_2 = p_- + p_+) \;+\; k_1k_2\\
&=\; k_1k_2 \;-\; p_-p_+ \;=\; \tfrac{1}{2}s \;-\; \tfrac{1}{2}(s-2m^2) \;=\; m^2.
\end{aligned} \tag{25}$$

Consequently,

$$
\begin{aligned}
A_{12} \;&=\; 4m^2 \times \Big(-m^2 \;-\; \tfrac{1}{2}(t+m^2) \;-\; \tfrac{1}{2}(u+m^2) \;+\; \tfrac{1}{2}(s-2m^2)\Big) \\
&\qquad -\; 4m^2(s-2m^2) \;-\; 2m^2(t+m^2) \;-\; 2m^2(u+m^2) \;+\; 8m^4 \\
&=\; 4m^2 \times \Big(+m^2 \;-\; (t+m^2) \;-\; (u+m^2) \;-\; \tfrac{1}{2}(s-2m^2)\Big) \\
&=\; -2m^2 \times (2t+2u+s) \\
&=\; -2m^2 \times (t+u+2m^2) \\
&=\; -2m^2(t-m^2) \;-\; 2m^2(u-m^2) \;-\; 8m^4.
\end{aligned}
\tag{26}
$$

Annihilation Summary

Having worked out the traces, let's plug them into eq. (13):

$$
\begin{aligned}
\overline{|\mathcal{M}|^2} \;&=\; \frac{e^4}{(t-m^2)^2} \times \Big(2(t-m^2)(u-3m^2) \;-\; 8m^4\Big) \\
&\quad +\; \frac{e^4}{(u-m^2)^2} \times \Big(2(u-m^2)(t-3m^2) \;-\; 8m^4\Big) \\
&\quad +\; \frac{2e^4}{(t-m^2)(u-m^2)} \times \Big(-2m^2(t-m^2) \;-\; 2m^2(u-m^2) \;-\; 8m^4\Big) \\
&=\; 2e^4 \left(\frac{u-3m^2}{t-m^2} \;+\; \frac{t-3m^2}{u-m^2} - \frac{2m^2}{u-m^2} \;-\; \frac{2m^2}{t-m^2}\right) \\
&\quad -\; 8e^4m^4 \left(\frac{1}{t-m^2} \;+\; \frac{1}{u-m^2}\right)^2 \\
&=\; 2e^4 \left[\frac{u-m^2}{t-m^2} \;+\; \frac{t-m^2}{u-m^2} \;-\; 4m^2 \left(\frac{1}{t-m^2} \;+\; \frac{1}{u-m^2}\right)\right. \\
&\qquad\qquad\qquad\qquad \left. -\; 4m^4 \left(\frac{1}{t-m^2} \;+\; \frac{1}{u-m^2}\right)^2\right],
\end{aligned}
\tag{27}
$$

or more compactly

$$
\overline{|\mathcal{M}|^2} \;=\; 2e^4 \left[\frac{u-m^2}{t-m^2} \;+\; \frac{t-m^2}{u-m^2} \;+\; 1 \;-\; \left(1 \;+\; \frac{2m^2}{t-m^2} \;+\; \frac{2m^2}{u-m^2}\right)^2\right]. \tag{28}
$$

This is our final result; the rest is kinematics.

Annihilation Kinematics

In the center of mass frame, $p^\mu_\mp = (E, \pm\mathbf{p})$ where $E = +\sqrt{\mathbf{p}^2 + m^2}$, and $k^\mu_{1,2} = (\omega, \pm\mathbf{k})$ where $\omega = |\mathbf{k}| = E$. Consequently,

$$\begin{aligned} s &= 4E^2, \\ t &= -(\mathbf{p}-\mathbf{k})^2 = -\mathbf{p}^2 - E^2 + 2|\mathbf{p}|E\cos\theta, \\ u &= -(\mathbf{p}+\mathbf{k})^2 = -\mathbf{p}^2 - E^2 - 2|\mathbf{p}|E\cos\theta, \\ t - m^2 &= -2E(E - |\mathbf{p}|\cos\theta), \\ u - m^2 &= -2E(E + |\mathbf{p}|\cos\theta), \end{aligned} \tag{29}$$

and therefore

$$\begin{aligned} \frac{u-m^2}{t-m^2} + \frac{t-m^2}{u-m^2} + 1 &= \frac{E+|\mathbf{p}|\cos\theta}{E-|\mathbf{p}|\cos\theta} + \frac{E-|\mathbf{p}|\cos\theta}{E+|\mathbf{p}|\cos\theta} + 1 \\ &= \frac{3E^2 + \mathbf{p}^2\cos^2\theta}{E^2 - \mathbf{p}^2\cos^2\theta} \\ &= \frac{3m^2 + \mathbf{p}^2(3+\cos^2\theta)}{m^2 + \mathbf{p}^2\sin^2\theta}, \\ \frac{1}{t-m^2} + \frac{1}{u-m^2} &= \frac{-1}{2E}\left(\frac{1}{E-|\mathbf{p}|\cos\theta} + \frac{1}{E+|\mathbf{p}|\cos\theta}\right) \\ &= \frac{-1}{2E} \times \frac{2E}{E^2 - \mathbf{p}^2\cos^2\theta} = \frac{-1}{m^2 + \mathbf{p}^2\sin^2\theta}, \\ 1 + \frac{2m^2}{t-m^2} + \frac{2m^2}{u-m^2} &= \frac{\mathbf{p}^2\sin^2\theta - m^2}{\mathbf{p}^2\sin^2\theta + m^2}. \end{aligned} \tag{30}$$

Thus

$$\overline{|\mathcal{M}|^2} = 2e^4\left[\frac{3m^2 + \mathbf{p}^2(3+\cos^2\theta)}{m^2 + \mathbf{p}^2\sin^2\theta} - \left(\frac{\mathbf{p}^2\sin^2\theta - m^2}{\mathbf{p}^2\sin^2\theta + m^2}\right)^2\right], \tag{31}$$

and finally the partial cross section of annihilation

$$\frac{d\sigma(e^+e^- \to \gamma\gamma)}{d\Omega_{\text{c.m.}}} = \frac{|\mathbf{k}|}{|\mathbf{p}|}\frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} = \frac{\alpha^2}{8E|\mathbf{p}|} \times \left[\frac{3m^2 + \mathbf{p}^2(3+\cos^2\theta)}{m^2 + \mathbf{p}^2\sin^2\theta} - \left(\frac{\mathbf{p}^2\sin^2\theta - m^2}{\mathbf{p}^2\sin^2\theta + m^2}\right)^2\right]. \tag{32}$$

For the non-relativistic electron and positron with $|\mathbf{p}| \ll m$, the expression in the square brackets becomes $3 - (-1)^2 = 2$, hence *isotropic* partial cross section

$$\frac{d\sigma(\text{slow } e^+e^- \to \gamma\gamma)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4m|\mathbf{p}|}\,. \tag{33}$$

And the total cross section in this limit is

$$\sigma_{\text{tot}}(\text{slow } e^+e^- \to \gamma\gamma) = \frac{4\pi}{2} \times \frac{\alpha^2}{4m|\mathbf{p}|} = \frac{\pi\alpha^2}{2m|\mathbf{p}|}\,, \tag{34}$$

where total solid angle is $4\pi/2$ because of 2 identical photons in the final state.

In the opposite limit of ultra-relativistic e^- and e^+ with $|\mathbf{p}| \approx E \gg m$, we have

$$\Big[\cdots\Big] \approx \frac{3+\cos^2\theta}{\sin^2\theta} - 1 = \frac{2(1+\cos^2\theta)}{\sin^2\theta} \tag{35}$$

and hence highly un-isotropic cross section

$$\frac{d\sigma(\text{fast } e^+e^- \to \gamma\gamma)}{d\Omega_{\text{c.m.}}} \approx \frac{\alpha^2}{4E^2} \times \frac{1+\cos^2\theta}{\sin^2\theta}\,. \tag{36}$$

Note how this cross-section is strongly peaked in the forward direction $\theta = 0$ where one photon continues the electron's motion while the other continues the positron's motion.

According to eq. (36), the total annihilation cross-section

$$\sigma_{\text{tot}}(\text{fast } e^+e^- \to \gamma\gamma) = 2\pi \int_0^{\pi/2} d\theta\, \sin\theta\, \frac{d\sigma}{d\Omega_{\text{cm}}} \tag{37}$$

diverges at small angles, but that's an artefact of the approximation (35) becoming inaccurate at small angles where $\mathbf{p}^2 \sin^2\theta \lesssim m^2$. Instead, for small angles we have

$$\left[\cdots\right] \;=\; \frac{4\mathbf{p}^2}{m^2+\mathbf{p}^2\theta^2} \;+\; O(1) \tag{38}$$

and consequently

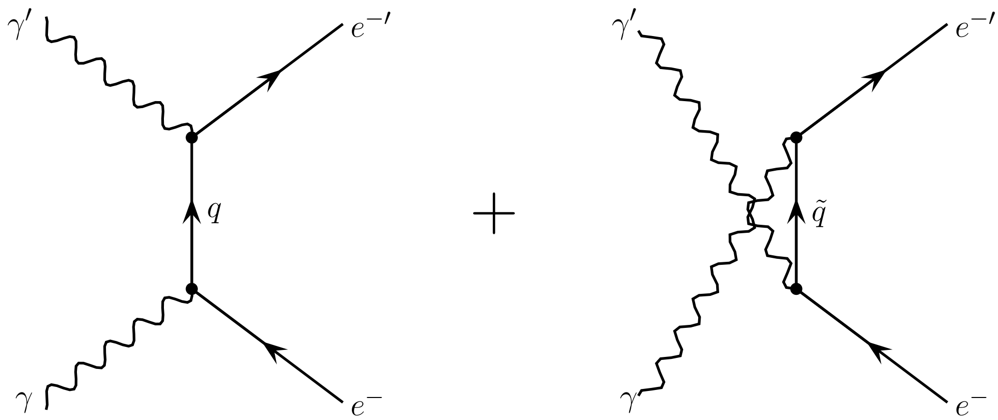
$$\frac{d\sigma(\text{fast } e^+e^- \to \gamma\gamma)}{d\Omega_{\text{c.m.}}} \;\approx\; \frac{\alpha^2}{4E^2} \times \frac{2\mathbf{p}^2}{m^2+\mathbf{p}^2\theta^2}\,. \tag{39}$$

This cross-section is strongly peaked in the forward direction, but it does not diverge. Instead,

$$\sigma_{\text{tot}}(\text{fast } e^+e^- \to \gamma\gamma) \;=\; \frac{\pi\alpha^2}{E^2} \times \left(\log\frac{2E}{m} - \frac{1}{2}\right). \tag{40}$$

Compton Scattering

Compton scattering of an electron and a photon $e^-\gamma \to e^-\gamma$ is related by crossing symmetry to the $e^-e^+ \to \gamma\gamma$ annihilation. Indeed, at the tree level there are two diagrams



(41)

which are obviously related by $s \leftrightarrow t$ crossing to the annihilation diagrams (1). Hence, given eq. (28) for the annihilation, we may immediately write down a similar formula for the Compton scattering without doing any work. All we need is to exchange $s \leftrightarrow t$ in eq. (28) and change the overall sign because we cross one fermion, thus

$$\overline{|\mathcal{M}^{\text{Compton}}|^2} \;=\; 2e^4\left[-\frac{u-m^2}{s-m^2} - \frac{s-m^2}{u-m^2} - 1 + \left(1 + \frac{2m^2}{s-m^2} + \frac{2m^2}{u-m^2}\right)^2\right]. \tag{42}$$

This is it; all we need to do now is kinematics.

Compton scattering is usually studied in the lab frame where the initial electron is at rest, $p^\mu = (m, \mathbf{0})$. In this frame, the initial and the final photon energies ω and ω' are related to photon's scattering angle θ via Compton's formula

$$\frac{1}{\omega'} = \frac{1}{\omega} + \frac{1-\cos\theta}{m_e}, \tag{43}$$

originally written by Arthur Compton in terms of the photon's wavelengths as

$$\lambda' - \lambda = \frac{h}{m_e c} \times (1-\cos\theta). \tag{44}$$

According to this formula, there is an upper limit on the energy of the final photon for any *fixed* $\theta \neq 0$: regardless of the initial energy ω, the final energy ω' can never exceed $m_e/(1-\cos\theta)$.

The Compton's formula follows from the energy–momentum conservation

$$\omega' + E' = \omega + m \quad \text{and} \quad \mathbf{k}' + \mathbf{p}' = \mathbf{k} + \mathbf{0}, \tag{45}$$

which imply

$$\mathbf{p}'^2 = (\mathbf{k}-\mathbf{k}')^2 = \mathbf{k}^2 + \mathbf{k}'^2 - 2\mathbf{k}\cdot\mathbf{k}' = \omega^2 + \omega'^2 - 2\omega\omega'\cos\theta \tag{46}$$

while

$$\mathbf{p}'^2 + m^2 = E'^2 = (\omega + m - \omega')^2 = \omega^2 + \omega'^2 - 2\omega\omega' + 2\omega m - 2\omega' m + m^2. \tag{47}$$

Subtracting these two formulae and canceling similar terms gives us

$$2\omega m = 2\omega' m + 2\omega\omega' \times (1-\cos\theta) \tag{48}$$

and hence eq. (43).

The Mandelstam variables s and u in the lab frame are

$$\begin{aligned} s &\equiv (k+p)^2 = (\omega+m)^2 - (\mathbf{k}+\mathbf{0})^2 = 2\omega m + m^2, \\ u &\equiv (k'-p)^2 = (\omega'-m)^2 - (\mathbf{k}'-\mathbf{0})^2 = -2\omega' m + m^2, \end{aligned} \tag{49}$$

and hence

$$s - m^2 = +2m\omega, \qquad u - m^2 = -2m\omega'. \tag{50}$$

Plugging these values into eq. (42), we have

$$\begin{aligned} -\frac{u-m^2}{s-m^2} - \frac{s-m^2}{u-m^2} - 1 &= +\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - 1, \\ 1 + \frac{2m^2}{s-m^2} + \frac{2m^2}{u-m^2} &= 1 + \frac{m}{\omega} - \frac{m}{\omega'} \\ &= -\cos\theta \end{aligned} \tag{51}$$

where the last equality follows from eq. (43), and therefore

$$\overline{|\mathcal{M}^{\text{Compton}}|^2} = 2e^4 \times \left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - 1 + \cos^2\theta\right). \tag{52}$$

Finally, we need the phase space factor for the lab frame. For a generic $2 \to 2$ scattering process,

$$\begin{aligned} d\sigma &= \overline{|\mathcal{M}|^2} \times d\mathcal{P}, \quad \text{where} \\ d\mathcal{P} &= \frac{1}{2E_1 \times 2E_2 \times \Delta v} \times \frac{d^3\mathbf{p}'_1}{(2\pi)^3\, 2E'_1} \times \frac{d^3\mathbf{p}'_3}{(2\pi)^2\, 2E'_2} \times (2\pi)^4 \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2) \\ &= \frac{1}{64\pi^2\, E_1 E_2 E'_1 E'_2\, \Delta v} \times d^3\mathbf{p}'_1\ \delta(E'_1 + E'_2 - E_1 - E_2)\Big|_{\mathbf{p}'_2 = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1} \\ &= \frac{d\Omega'_1}{64\pi^2} \times \frac{\mathbf{p}'^2_1}{E_1 E_2 E'_1 E'_2\, \Delta v} \times \left(\frac{d(E'_1 + E'_2)}{d|\mathbf{p}'_1|}\bigg|_{\mathbf{p}'_2 = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1} \right)^{-1}. \end{aligned} \tag{53}$$

Specializing to the Compton scattering and the lab frame for initial electron, we immediately obtain

$$d\mathcal{P} = \frac{d\Omega_\gamma}{64\pi^2} \times \frac{\omega'^2}{\omega m \omega' E'} \times \left(1 + \frac{dE'}{d\omega'}\bigg|_{\mathbf{p}' = \mathbf{k} - \mathbf{k}'}\right)^{-1}. \tag{54}$$

The only non-trivial issue here is the derivative in the parentheses. This derivative should be taken for a fixed photon angle θ and before applying the energy conservation rule

$E'_e = \omega + m - \omega'$. Instead, we use the momentum conservation $\mathbf{p}' = \mathbf{k} - \mathbf{k}'$, hence eq. (46) for the $\mathbf{p}'^2$ and consequently

$$E'^2 \;=\; \mathbf{p}'^2 \;+\; m^2 \;=\; \omega^2 \;+\; \omega'^2 \;-\; 2\omega\omega' \times \cos\theta + \; m^2. \tag{55}$$

For fixed ω and θ,

$$2E' \times dE' \;=\; 2|\mathbf{p}'| \times d|\mathbf{p}'| \;=\; 2(\omega' - \omega\cos\theta) \times d\omega', \tag{56}$$

and hence

$$\frac{dE'}{d\omega'} \;=\; \frac{\omega' - \omega\cos\theta}{E'}\,. \tag{57}$$

Once we have taken this derivative, me may now use energy conservation, thus

$$1 \;+\; \frac{dE'}{d\omega'} \;=\; \frac{E' + \omega' - \omega\cos\theta}{E'} \;=\; \frac{m + \omega - \omega\cos\theta}{E'} \;=\; \frac{\omega m}{\omega' E'}\,, \tag{58}$$

where the last equality follows from the Compton formula (43). Plugging the derivative (58) into eq. (54), we arrive at

$$d\mathcal{P} \;=\; \frac{d\Omega_\gamma}{64\pi^2} \times \frac{\omega'^2}{m^2\omega^2} \tag{59}$$

and hence the *Klein–Nishina formula* for the partial cross-section:

$$\frac{d\sigma^{\text{Compton}}}{d\Omega_{\text{lab}}} \;=\; \frac{\alpha^2}{2m_e^2} \times \frac{\omega'^2}{\omega^2} \times \left(\frac{\omega'}{\omega} \;+\; \frac{\omega}{\omega'} \;-\; \sin^2\theta\right) \tag{60}$$

where ω' is given by eq. (43).

For low photon energies $\omega \ll m_e$, the Compton's formula gives $\omega' \approx \omega$, and the Klein–Nishina cross-section (60) becomes the good old Thompson cross-section

$$\frac{d\sigma^{\text{Compton}}}{d\Omega_{\text{lab}}} \;\to\; \frac{d\sigma^{\text{Thompson}}}{d\Omega_{\text{lab}}} \;=\; \frac{\alpha^2}{2m_e^2} \times (2 - \sin^2\theta \;=\; 1 + \cos^2\theta), \tag{61}$$

and the total cross-section is

$$\sigma_{\text{total}}^{\text{Thompson}} \;=\; \frac{8\pi}{3}\,\frac{\alpha^2}{m_e^2} \;\approx\; 0.663\,\text{barn}. \tag{62}$$

On the other hand, for very high photon energies $\omega \gg m_e$ and $\theta \not\approx 0$, we have

$$\omega' \ll \omega \quad\Longrightarrow\quad \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta \;\approx\; \frac{\omega}{\omega'}\,, \tag{63}$$

and the Klein–Nishina formula becomes

$$\frac{d\sigma^{\text{Compton}}}{d\Omega_{\text{lab}}} \;\approx\; \frac{\alpha^2}{2m_e^2}\times\frac{\omega'}{\omega} \;\approx\; \frac{\alpha^2}{2m_e\times\omega}\times\frac{1}{1-\cos\theta}\,. \tag{64}$$

This approximation is not accurate at small angles $\theta \lesssim \sqrt{2m_e/\omega}$ for which $\omega' \not\ll \omega$, so the cross section does not really diverge for $\theta \to 0$. Instead, at small angles we have large but finite partial cross-section

$$\frac{d\sigma^{\text{Compton}}}{d\Omega_{\text{lab}}} \approx \frac{\alpha^2}{m_e\times\omega}\times\frac{\theta^4 - 2\theta^2(2m_e/\omega) + 2(2m_e/\omega)^2}{(\theta^2 + (2m_e/\omega))^3} \not\to \infty \tag{65}$$

and hence finite total cross-section

$$\sigma_{\text{total}}^{\text{Compton}} \;\approx\; \frac{\pi\alpha^2}{m_e\times\omega}\times\left(\log\frac{2\omega}{m_e} + \frac{1}{2}\right). \tag{66}$$