QFT Dimensional Analysis

In $\hbar = c = 1$ units, all quantities are measured in units of energy to some power. For example $\big[m\big] = \big[p^\mu\big] = E^{+1}$ while $\big[x^\mu\big] = E^{-1}$ where $[m]$ stands for the *dimensionality* of the mass rather than the mass itself, and ditto for the $\big[p^\mu\big]$, $\big[x^\mu\big]$, *etc.* The action

$$S = \int d^4x\,\mathcal{L}$$

is dimensionless (in $\hbar \neq 1$ units, $\big[S\big] = \hbar$), so the Lagrangian of a 4D field theory has dimensionality $[\mathcal{L}] = E^{+4}$.

Canonical dimensions of quantum fields follow from the free-field Lagrangians. A scalar field $\Phi(x)$ has

$$\mathcal{L}_{\text{free}} = \tfrac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi - \tfrac{1}{2}m^2\Phi^2, \tag{1}$$

so $\big[\mathcal{L}\big] = E^{+4}$, $\big[m^2\big] = E^{+2}$, and $\big[\partial_\mu\big] = E^{+1}$ imply $\big[\Phi\big] = E^{+1}$. Likewise, the EM field has

$$\mathcal{L}^{\text{EM}}_{\text{free}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad\Longrightarrow\quad \big[F_{\mu\nu}\big] = E^{+2}, \tag{2}$$

and since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the $A_\nu(x)$ field has dimension

$$\big[A_\nu\big] = \big[F_{\mu\nu}\big]\,/\,\big[\partial_\mu\big] = E^{+1}. \tag{3}$$

The massive vector fields also have $\big[A_\nu\big] = E^{+1}$ so that both terms in

$$\mathcal{L}_{\text{free}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m^2A_\nu A^\nu \tag{4}$$

have dimensions $\big[F^2\big] = \big[m^2A^2\big] = E^{+4}$.

In fact, all *bosonic* fields in 4D spacetime have canonical dimensions E^{+1} because their kinetic terms are quadratic in ∂_μ(field). On the other hand, the fermionic fields like the Dirac field $\Psi(x)$ with free Lagrangian

$$\mathcal{L}_{\text{free}} = \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \tag{5}$$

have kinetic terms with two fields but only one ∂_μ. Consequently, $\big[\mathcal{L}\big] = E^{+4}$ implies $\big[\overline{\Psi}\Psi\big] = E^{+3}$ and hence $\big[\Psi\big] = \big[\overline{\Psi}\big] = E^{+3/2}$. Similarly, all other types of fermionic fields in 4D have canonical dimension $E^{+3/2}$.

In QFTs in other spacetime dimensions $d \neq 4$, the bosonic fields such as scalars and vectors have canonical dimension

$$\big[\Phi\big] \;=\; \big[A_\nu\big] \;=\; E^{+(d-2)/2} \tag{6}$$

while the fermionic fields have canonical dimension

$$\big[\Psi\big] \;=\; E^{+(d-1)/2}. \tag{7}$$

In perturbation theory, dimensionality of coupling parameters such as λ in $\lambda\Phi^4$ theory or e in QED follows from the field's canonical dimensions. For example, in a 4D scalar theory with Lagrangian

$$\mathcal{L} \;=\; \tfrac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi \;-\; \tfrac{1}{2}m^2\Phi^2 \;-\sum_{n\geq 3}\frac{C_n}{n!}\,\Phi^n, \tag{8}$$

the coupling C_n of the Φ^n term has dimensionality

$$\big[C_n\big] \;=\; \big[\mathcal{L}\big]\Big/\big[\Phi\big]^n \;=\; E^{4-n}. \tag{9}$$

In particular, the cubic coupling C_3 has positive energy dimension E^{+1}, the quartic coupling $\lambda = C_4$ is dimensionless, while all the higher-power couplings have negative energy dimensions E^{negative}.

Now consider a theory with a single coupling g of dimensionality $\big[g\big] = E^\Delta$. The perturbation theory in g amounts to a power series expansion

$$\mathcal{M}(\text{momenta}, g) \;=\; \sum_N \left(\frac{g}{\mathcal{E}^\Delta}\right)^N \times F_N(\text{momenta}) \tag{10}$$

where $\mathcal{E}$ is the overall energy scale of the process in question and all the F_N functions of momenta have the same dimensionality. The power series (10) is asymptotic rather than convergent, so it makes sense only when the expansion parameter is small,

$$\frac{g}{\mathcal{E}^\Delta} \;\ll\; 1. \tag{11}$$

For a dimensionless coupling g, this condition is simply $g \ll 1$, but for $\Delta \neq 0$, the situation is more complicated.

For couplings of positive dimensionality $\Delta > 0$, the expansion parameter (11) is always small for for high-energy processes with $\mathcal{E} \gg g^{1/\Delta}$. But for low energies $\mathcal{E} \lesssim g^{1/\Delta}$, the expansion parameter becomes large and the perturbation theory breaks down. This is a major problem for theories with $\Delta > 0$ couplings of *massless particles.* However, if all the particles participating in a $\Delta > 0$ coupling are massive, then all processes have energies $\mathcal{E} \gtrsim M_{\text{lightest}}$, and this makes couplings with $\Delta > 0$ OK as long as

$$g \ll M_{\text{lightest}}^{\Delta} . \tag{12}$$

Couplings of negative dimensionality $\Delta < 0$ have an opposite problem: The expansion parameter (11) is small at low energies but becomes large at high energies $\mathcal{E} \gtrsim g^{-1/\Delta}$. Beyond the maximal energy

$$E^{\max} \sim g^{1/(-\Delta)}, \tag{13}$$

the perturbation theory breaks down and we may no longer compute the S–matrix elements $\mathcal{M}$ using any finite number of Feynman diagrams.

Worse, in Feynman diagrams with loops one must worry not only about momenta k^μ of the incoming and outgoing particles but also about momenta q^μ of the internal lines. Basically, an L–loop diagram contributing to N^{th} term in the expansion (10) produces something like

$$g^N \times \int d^{4L}q\, \mathcal{F}_N(q,k,m) \qquad \text{where} \quad \big[\mathcal{F}_N\big] = E^{-N\Delta-4L+\mathrm{C}}, \quad C = \text{const.} \tag{14}$$

For very large loop momenta $q \gg k, m$, dimensionality implies $\mathcal{F}_N \propto q^{-N\Delta-4L+\mathrm{C}}$, so for $N(-\Delta) + C \geq 0$, the integral (14) diverges as $q \to \infty$. Moreover, the degree of divergence increases with the order N of the perturbation theory, so any scattering amplitude becomes divergent at high orders. Therefore, *field theories with $\Delta < 0$ couplings do not work as complete theories.*

However, theories with $\Delta < 0$ may be used as approximate *effective theories* (without the divergent loop graphs) for low-energy processes, $\mathcal{E} \lesssim \Lambda$ for some $\Lambda < g^{-1/\Delta}$. For example, the Fermi theory of weak interactions

$$\mathcal{L}_{\rm int} \;=\; \frac{G_F}{\sqrt{2}} \sum_{\substack{\text{appropriate}\\ \text{fermions}}} \overline{\Psi}\gamma_\mu(1-\gamma^5)\Psi \times \overline{\Psi}\gamma^\mu(1-\gamma^5)\Psi \tag{15}$$

has coupling G_F of dimension $\big[G_G\big] = E^{-2}$; its value is $G_F \approx 1.17\cdot 10^{-5}\,\text{GeV}^{-2}$. This is a good effective theory for low-energy weak interactions, but it cannot be used for energies $\mathcal{E} \gtrsim 1/\sqrt{G_F} \sim 300$ GeV, not even theoretically. In real life, the Fermi theory works only for $\mathcal{E} \ll M_W \sim 80$ GeV; at higher energies, one should use the complete $SU(2)\times U(1)$ electroweak theory instead of the Fermi theory.

Similar to the Fermi theory, most effective theories with $\Delta < 0$ couplings are low-energy limits of more complicated theories with extra heavy particles of masses $M \lesssim g^{-1/\Delta}$ but no $\Delta < 0$ couplings.

In QFTs which are valid for all energies, all coupling must have zero or positive energy dimensions. In 4D, a coupling involving b bosonic fields (scalar or vector), f fermionic fields, and δ derivatives ∂_μ has dimensionality

$$\Delta \;=\; 4 \;-\; b \;-\; \tfrac{3}{2}f \;-\; \delta. \tag{16}$$

Thus, only the boson^3 couplings have $\Delta > 0$ while the $\Delta = 0$ couplings comprise boson^4, boson $\times$ fermion^2, and boson^2 $\times$ ∂boson. All other coupling types have $\Delta < 0$ and are not allowed (except in effective theories).

Here is the complete list of the allowed couplings in 4D.

1. Scalar couplings

$$-\frac{\mu}{3!}\,\Phi^3 \quad\text{and}\quad -\frac{\lambda}{4!}\,\Phi^4. \tag{17}$$

   Note: the higher powers Φ^5, Φ^6, *etc.*, are not allowed because the couplings would have $\Delta < 0$.

2. Gauge couplings of vectors to charged scalars

$$-iqA^\mu\left(\Phi^*\partial_\mu\Phi\;-\;\Phi\partial_\mu\Phi^*\right)\;+\;q^2\Phi^*\Phi A_\mu A^\mu\;\subset\;D_\mu\Phi^*\,D^\mu\Phi. \tag{18}$$

3. Non-abelian gauge couplings between the vector fields

$$-gf^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c}\;-\;\frac{g^2}{4}\,f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e}\;\subset\;-\frac{1}{4}\,F^a_{\mu\nu}F^{\mu\nu a}. \tag{19}$$

4. Gauge couplings of vectors to charged fermions,

$$-qA^\mu\times\overline{\Psi}\gamma_\mu\Psi\;\subset\;\overline{\Psi}(i\gamma_\mu D^\mu)\Psi. \tag{20}$$

If the fermions are massless and chiral, we may also have

$$-qA_\mu\times\overline{\Psi}\gamma^\mu\,\frac{1\mp\gamma^5}{2}\,\Psi, \tag{21}$$

or in the Weyl fermion language

$$-qA_\mu\times\chi^\dagger\bar{\sigma}_\mu\chi.$$

5. Yukawa couplings of scalars to fermions,

$$-y\Phi\times\overline{\Psi}\Psi\quad\text{or}\quad-\,iy\Phi\times\overline{\Psi}\gamma^5\Psi. \tag{22}$$

If parity is conserved, in the first terms Φ should be a true scalar, and in the second term a pseudo-scalar.

In other spacetime dimensions $d\neq 3+1$, a coupling involving b bosonic fields, f fermionic fields, and δ derivatives has energy dimension

$$\Delta\;=\;d\;-\;b\times\frac{d-2}{2}\;-\;f\times\frac{d-1}{2}\;-\;\delta\;=\;b\;+\;\tfrac{1}{2}f\;-\;\delta\;-\;\frac{b+f-2}{2}\times d. \tag{23}$$

Since all interactions involve three or more fields, thus $b+f\geq 3$, the dimensionality of any particular coupling always decreases with d. Consequently, there are more perturbatively-allowed couplings with $\Delta\geq 0$ in lower dimensions $d=2+1$ or $d=1+1$ but fewer allowed couplings in higher dimensions $d>3+1$. In particular,

- *In $d \geq 6+1$ dimensions all couplings have $\Delta < 0$ and there are no UV-complete quantum field theories,* or at least no perturbative UV-complete quantum field theories.
- In $d = 5+1$ dimensions there is a unique $\Delta = 0$ coupling $(\mu/6)\Phi^3$, while all the other couplings have $\Delta < 0$. Consequently, the only perturbative UV-complete theories are scalar theories with cubic potentials,

$$\mathcal{L} \;=\; \sum_a \left(\tfrac{1}{2}(\partial_\mu\Phi_a)^2 \;-\; \tfrac{1}{2}m_a^2\Phi_a^2\right) \;-\; \tfrac{1}{6}\sum_{a,b,c}\mu_{abc}\Phi_a\Phi_b\Phi_c\,. \tag{24}$$

  However, while such theories are perturbatively OK, they do not have stable vacua. Indeed, a cubic potential is un-bounded from below — it goes to $-\infty$ along half of the directions in the field space — so even if it has a *local* minimum at $\Phi_a = 0$, it's not the global minimum. Consequently, in the quantum theory, the naive vacuum with $\langle\Phi_a\rangle = 0$ would decay by tunneling to a run-away state with $\langle\Phi_a\rangle \to \pm\infty$.
- In $d = 4+1$ dimensions, the $(\mu/6)\Phi^3$ coupling has positive $\Delta = +\frac{1}{2}$ while all the other couplings have negative energy dimensions. Again, the only perturbative UV-complete theories are scalar theories with cubic potentials, but they do not have stable vacua.

⋆ The bottom line is, *in $d > 3+1$ dimensions, all quantum field theories are effective theories* for low-enough energies. At higher energies, a different kind of theory must take over — perhaps a theory in a discrete space, perhaps a string theory, or maybe something more exotic.

On the other hand, in lower dimensions $d = 2+1$ or $d = 1+1$ there are a lot of allowed couplings with $\Delta \geq 0$. In particular, in $d = 2+1$ dimensions the allowed couplings include:

- ○ Scalar couplings $(C_n/n!)\Phi^n$ up to $n = 6$;
- ○ gauge and Yukawa couplings like in 4D;
- ○ Yukawa-like couplings $\tilde{y}\Phi^2 \times \overline{\Psi}\Psi$ involving 2 scalars;
- ○ gauge-like couplings with $g_{\rm gauge}$ linearly dependent on a neutral scalar field:

$$D_\mu\Psi \;=\; \partial_\mu\Psi \;+\; i(g_0 + c\phi)A_\mu\Psi \quad\Longrightarrow\quad \overline{\Psi}(i\gamma^\mu D_\mu)\Psi \;\supset\; -c\phi A_\mu\,\overline{\Psi}\gamma^\mu\Psi, \tag{25}$$

and likewise

$$D^\mu\Phi^* D_\mu\Phi \;\supset\; -ic\phi A^\mu \times (\Phi^*\partial_\mu\Phi \,-\, \Phi\partial_\mu\Phi^*) \;+\; c^2\phi^2 \times \Phi^*\Phi A_\mu A^\mu, \tag{26}$$

or non-abelian

$$-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} \;\supset\; -c\phi \times f^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} \;-\; \frac{c^2}{4}\phi^2 \times f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e}. \tag{27}$$

There are also combinations of $g = g_0 + c\phi$ with Chern–Simons couplings (like the 3D photon mass term in the mid-term exam and its non-abelian generalizations), but I don't want to get into their details here.

∗ There might be some other allowed couplings in 3D, but never mind for now.

Finally, *in $d = 1 + 1$ dimensions there is an infinite number of allowed $\Delta \geq 0$ couplings.* Indeed, for $d = 1+1$ the bosonic fields have energy dimension E^0, so Δ of a coupling does not depend on the number b of bosonic fields it involves but only on the numbers of derivatives and fermionic fields,

$$\Delta \;=\; 2 - \;\delta \;-\; \tfrac{1}{2}f. \tag{28}$$

Consequently, all scalar potentials $V(\Phi)$ — including $C_n\Phi^n$ terms for any n, and even the non-polynomial potentials — have $\Delta = +2$, so any $V(\Phi)$ is allowed in 2D. Likewise, all Yukawa-like couplings $\Phi^n\overline{\Psi}\Psi$ have $\Delta = +1$, so we may have terms like $y_{IJ}(\Phi) \times \overline{\Psi}^I\Psi^J$ for any functions $y_{IJ}(\Phi)$.

At the $\Delta = 0$ level, we are allowed generic Riemannian metrics $g_{ij}(\Phi)$ of the scalar field space, hence field-dependent kinetic terms

$$\mathcal{L}_{\rm kin} \;=\; \tfrac{1}{2}g_{ij}(\phi) \times \partial^\mu\phi^i\,\partial_\mu\phi^j, \tag{29}$$

as well as a whole bunch of fermionic terms with arbitrary scalar-dependent coefficients,

$$\begin{aligned}\mathcal{L}_\Psi \;\supset\; \tfrac{1}{4}g_{IJ}(\Phi) \times \overline{\Psi}^I\gamma^\mu\left(i\overrightarrow{\partial_\mu} - i\overleftarrow{\partial_\mu}\right)\Psi^J \;+\; \Gamma_{IJk}(\Phi) \times \partial_\mu\Phi^k \times \overline{\Psi}^I\gamma^\mu\Psi^J \\ +\; \tfrac{1}{2}R_{IJKL}(\Phi) \times \overline{\Psi}^I\gamma^\mu\Psi^J \times \overline{\Psi}^K\gamma_\mu\Psi^L.\end{aligned} \tag{30}$$

In addition, there are gauge couplings with arbitrary scalar dependent $g_{\rm gauge}(\Phi)$, chiral couplings to Weyl or Majorana-Weyl fermions, *etc.*, *etc.*.