QFT Dimensional Analysis

In $\hbar = c = 1$ units, all quantities are measured in units of energy to some power. For example $[m] = [p^\mu] = E^{+1}$ while $[x^\mu] = E^{-1}$ where $[m]$ stands for the dimensionality of the mass rather than the mass itself, and ditto for the $[p^\mu], [x^\mu]$ etc. The action

$$S = \int d^4x \mathcal{L}$$

is dimensionless (in $\hbar \neq 1$ units, $[S] = \hbar$), so the Lagrangian of a 4D field theory has dimensionality $[\mathcal{L}] = E^{+4}$.

Canonical dimensions of quantum fields follow from the free-field Lagrangians. A scalar field $\Phi(x)$ has

$$\mathcal{L}_{\text{free}} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2,$$

so $[\mathcal{L}] = E^{+4}$, $[m^2] = E^{+2}$, and $[\partial_\mu] = E^{+1}$ imply $[\Phi] = E^{+1}$. Likewise, the EM field has

$$\mathcal{L}_{\text{free}}^{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \implies [F_{\mu\nu}] = E^{+2},$$

and since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the $A_\nu(x)$ field has dimension

$$[A_\nu] = [F_{\mu\nu}] / [\partial_\mu] = E^{+1}.$$  

The massive vector fields also have $[A_\nu] = E^{+1}$ so that both terms in

$$\mathcal{L}_{\text{free}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\nu A^\nu$$

have dimensions $[F^2] = [m^2 A^2] = E^{+4}$.

In fact, all bosonic fields in 4D spacetime have canonical dimensions $E^{+1}$ because their kinetic terms are quadratic in $\partial_\mu$ (field). On the other hand, the fermionic fields like the Dirac field $\Psi(x)$ with free Lagrangian

$$\mathcal{L}_{\text{free}} = \bar{\Psi} (i \gamma^\mu \partial_\mu - m) \Psi$$

have kinetic terms with two fields but only one $\partial_\mu$. Consequently, $[\mathcal{L}] = E^{+4}$ implies $[\bar{\Psi} \Psi] = E^{+3}$ and hence $[\Psi] = [\bar{\Psi}] = E^{+3/2}$. Similarly, all other types of fermionic fields in 4D have canonical dimension $E^{+3/2}$. 

1
In QFTs in other spacetime dimensions \( d \neq 4 \), the bosonic fields such as scalars and vectors have canonical dimension

\[
[\Phi] = [A_\nu] = E^{+(d-2)/2}
\]  

while the fermionic fields have canonical dimension

\[
[\Psi] = E^{+(d-1)/2}.
\]  

In perturbation theory, dimensionality of coupling parameters such as \( \lambda \) in \( \lambda \Phi^4 \) theory or \( e \) in QED follows from the field’s canonical dimensions. For example, in a 4D scalar theory with Lagrangian

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \sum_{n \geq 3} \frac{C_n}{n!} \Phi^n,
\]  

the coupling \( C_n \) of the \( \Phi^n \) term has dimensionality

\[
[C_n] = [\mathcal{L}] / [\Phi]^n = E^{4-n}.
\]  

In particular, the cubic coupling \( C_3 \) has positive energy dimension \( E^1 \), the quartic coupling \( \lambda = C_4 \) is dimensionless, while all the higher-power couplings have negative energy dimensions \( E_{negative} \).

Now consider a theory with a single coupling \( g \) of dimensionality \( [g] = E^\Delta \). The perturbation theory in \( g \) amounts to a power series expansion

\[
\mathcal{M}(\text{momenta}, g) = \sum_N \left( \frac{g}{E^\Delta} \right)^N \times F_N(\text{momenta})
\]  

where \( \mathcal{E} \) is the overall energy scale of the process in question and all the \( F_N \) functions of momenta have the same dimensionality. The power series (10) is asymptotic rather than convergent, so it makes sense only when the expansion parameter is small,

\[
\frac{g}{E^\Delta} \ll 1.
\]  

For a dimensionless coupling \( g \), this condition is simply \( g \ll 1 \), but for \( \Delta \neq 0 \), the situation is more complicated.
For couplings of positive dimensionality $\Delta > 0$, the expansion parameter (11) is always small for high-energy processes with $\mathcal{E} \gg g^{1/\Delta}$. But for low energies $\mathcal{E} \lesssim g^{1/\Delta}$, the expansion parameter becomes large and the perturbation theory breaks down. This is a major problem for theories with $\Delta > 0$ couplings of massless particles. However, if all the particles participating in a $\Delta > 0$ coupling are massive, then all processes have energies $\mathcal{E} \gtrsim M_{\text{lightest}}$, and this makes couplings with $\Delta > 0$ OK as long as

$$g \ll M_{\text{lightest}}^{\Delta}. \quad (12)$$

Couplings of negative dimensionality $\Delta < 0$ have an opposite problem: The expansion parameter (11) is small at low energies but becomes large at high energies $\mathcal{E} \gtrsim g^{-1/\Delta}$. Beyond the maximal energy

$$E_{\text{max}}^{\Delta} \sim g^{1/(-\Delta)}, \quad (13)$$

the perturbation theory breaks down and we may no longer compute the S–matrix elements $\mathcal{M}$ using any finite number of Feynman diagrams.

Worse, in Feynman diagrams with loops one must worry not only about momenta $k^\mu$ of the incoming and outgoing particles but also about momenta $q^\mu$ of the internal lines. Basically, an $L$–loop diagram contributing to $N^{\text{th}}$ term in the expansion (10) produces something like

$$g^N \times \int d^{4L}q \mathcal{F}_N(q, k, m) \text{ where } \mathcal{F}_N = E^{-N\Delta - 4L + C}, \quad C = \text{const}. \quad (14)$$

For very large loop momenta $q \gg k, m$, dimensionality implies $\mathcal{F}_N \propto q^{-N\Delta - 4L + C}$, so for $N(-\Delta) + C \geq 0$, the integral (14) diverges as $q \to \infty$. Moreover, the degree of divergence increases with the order $N$ of the perturbation theory, so any scattering amplitude becomes divergent at high orders. Therefore, field theories with $\Delta < 0$ couplings do not work as complete theories.

However, theories with $\Delta < 0$ may be used as approximate effective theories (without the divergent loop graphs) for low-energy processes, $\mathcal{E} \lesssim \Lambda$ for some $\Lambda < g^{-1/\Delta}$. For example,
the Fermi theory of weak interactions

\[
\mathcal{L}_{\text{int}} = \frac{G_F}{\sqrt{2}} \sum_{\text{appropriate fermions}} \bar{\Psi} \gamma_\mu (1 - \gamma^5) \Psi \times \bar{\Psi} \gamma^\mu (1 - \gamma^5) \Psi
\]  

(15)

has coupling \( G_F \) of dimension \( [G_F] = E^{-2} \); its value is \( G_F \approx 1.17 \cdot 10^{-5} \text{GeV}^{-2} \). This is a good effective theory for low-energy weak interactions, but it cannot be used for energies \( \mathcal{E} \gtrsim 1/\sqrt{G_F} \sim 300 \text{GeV} \), not even theoretically. In real life, the Fermi theory works only for \( \mathcal{E} \ll M_W \sim 80 \text{GeV} \); at higher energies, one should use the complete \( SU(2) \times U(1) \) electroweak theory instead of the Fermi theory.

Similar to the Fermi theory, most effective theories with \( \Delta < 0 \) couplings are low-energy limits of more complicated theories with extra heavy particles of masses \( M \lesssim g^{-1/\Delta} \) but no \( \Delta < 0 \) couplings.

In QFTs which are valid for all energies, all coupling must have zero or positive energy dimensions. In 4D, a coupling involving \( b \) bosonic fields (scalar or vector), \( f \) fermionic fields, and \( \delta \) derivatives \( \partial_\mu \) has dimensionality

\[
\Delta = 4 - b - \frac{3}{2} f - \delta.
\]  

(16)

Thus, only the boson\(^3\) couplings have \( \Delta > 0 \) while the \( \Delta = 0 \) couplings comprise boson\(^4\), boson \( \times \) fermion\(^2\), and boson\(^2 \times \partial \) boson. All other coupling types have \( \Delta < 0 \) and are not allowed (except in effective theories).

Here is the complete list of the allowed couplings in 4D.

1. Scalar couplings

\[
- \frac{\mu}{3!} \Phi^3 \quad \text{and} \quad - \frac{\lambda}{4!} \Phi^4.
\]  

(17)

Note: the higher powers \( \Phi^5, \Phi^6, \text{etc.} \), are not allowed because the couplings would have \( \Delta < 0 \).
2. Gauge couplings of vectors to charged scalars

\[-iq^{\mu} (\Phi^* \partial_{\mu} \Phi - \Phi \partial_{\mu} \Phi^*) + q^2 \Phi^* \Phi A_{\mu} A^{\mu} \subset D_\mu \Phi^* D^\mu \Phi. \quad (18)\]

3. Non-abelian gauge couplings between the vector fields

\[-gf^{abc} (\partial_{\mu} A^a_{\nu}) A^{b}_{\mu} A^{c}_{\nu} - \frac{g^2}{4} f^{abc} f^{ade} A^b_{\mu} A^c_{\nu} A^{d}_{\mu} A^{e}_{\nu} \subset -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}. \quad (19)\]

4. Gauge couplings of vectors to charged fermions,

\[-q A^{\mu} \times \bar{\Psi} \gamma_\mu \Psi \subset \bar{\Psi} (i \gamma_\mu D^\mu) \Psi. \quad (20)\]

If the fermions are massless and chiral, we may also have

\[-q A^{\mu} \times \bar{\Psi} \gamma^5 \Psi_{\frac{1}{2}} \Psi. \quad (21)\]

or in the Weyl fermion language

\[-q A^{\mu} \times \chi^\dagger \bar{\sigma}_\mu \chi. \]

5. Yukawa couplings of scalars to fermions,

\[-y \Phi \times \bar{\Psi} \Psi \text{ or } -iy \Phi \times \bar{\Psi} \gamma^5 \Psi. \quad (22)\]

If parity is conserved, in the first terms $\Phi$ should be a true scalar, and in the second term a pseudo-scalar.

In other spacetime dimensions $d \neq 3+1$, a coupling involving $b$ bosonic fields, $f$ fermionic fields, and $\delta$ derivatives has energy dimension

\[\Delta = d - b \times \frac{d-2}{2} - f \times \frac{d-1}{2} - \delta = b + \frac{1}{2} f - \delta - \frac{b+f-2}{2} \times d. \quad (23)\]

Since all interactions involve three or more fields, thus $b + f \geq 3$, the dimensionality of any particular coupling always decreases with $d$. Consequently, there are more perturbatively-allowed couplings with $\Delta \geq 0$ in lower dimensions $d = 2 + 1$ or $d = 1 + 1$ but fewer allowed couplings in higher dimensions $d > 3 + 1$. In particular,
• In \( d \geq 6 + 1 \) dimensions all couplings have \( \Delta < 0 \) and there are no UV-complete quantum field theories, or at least no perturbative UV-complete quantum field theories.

• In \( d = 5 + 1 \) dimensions there is a unique \( \Delta = 0 \) coupling \( (\mu/6)\Phi^3 \), while all the other couplings have \( \Delta < 0 \). Consequently, the only perturbative UV-complete theories are scalar theories with cubic potentials,

\[
\mathcal{L} = \sum\limits_a \left( \frac{1}{2} (\partial_\mu \Phi_a)^2 - \frac{1}{2} m_a^2 \Phi_a^2 \right) - \frac{1}{6} \sum\limits_{a,b,c} \mu_{abc} \Phi_a \Phi_b \Phi_c . \tag{24}
\]

However, while such theories are perturbatively OK, they do not have stable vacua. Indeed, a cubic potential is un-bounded from below — it goes to \( -\infty \) along half of the directions in the field space — so even if it has a local minimum at \( \Phi_a = 0 \), it’s not the global minimum. Consequently, in the quantum theory, the naive vacuum with \( \langle \Phi_a \rangle = 0 \) would decay by tunneling to a run-away state with \( \langle \Phi_a \rangle \to \pm \infty \).

• In \( d = 4 + 1 \) dimensions, the \( (\mu/6)\Phi^3 \) coupling has positive \( \Delta = +\frac{1}{2} \) while all the other couplings have negative energy dimensions. Again, the only perturbative UV-complete theories are scalar theories with cubic potentials, but they do not have stable vacua.

★ The bottom line is, in \( d > 3 + 1 \) dimensions, all quantum field theories are effective theories for low-enough energies. At higher energies, a different kind of theory must take over — perhaps a theory in a discrete space, perhaps a string theory, or maybe something more exotic.

On the other hand, in lower dimensions \( d = 2 + 1 \) or \( d = 1 + 1 \) there are a lot of allowed couplings with \( \Delta \geq 0 \). In particular, in \( d = 2 + 1 \) dimensions the allowed couplings include:

– Scalar couplings \((C_n/n!)\Phi^n\) up to \( n = 6 \);

– gauge and Yukawa couplings like in 4D;

– Yukawa-like couplings \( \tilde{y} \Phi^2 \times \overline{\Psi} \Psi \) involving 2 scalars;

– gauge-like couplings with \( g_{\text{gauge}} \) linearly dependent on a neutral scalar field:

\[
D_\mu \Psi = \partial_\mu \Psi + i(g_0 + c\phi) A_\mu \Psi \implies \overline{\Psi} (i\gamma^\mu D_\mu) \Psi \supset -c\phi A_\mu \overline{\Psi} \gamma^\mu \Psi , \tag{25}
\]
and likewise

\[ D^\mu \Phi^* D_\mu \Phi \supset -ic\phi A^\mu \times (\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*) + c^2 \phi^2 \times \Phi^* \Phi A_\mu A^\mu, \]  

(26)
or non-abelian

\[-\frac{1}{4} F_{\mu \nu} F^{\mu \nu} \supset -c\phi \times f^{abc}(\partial_\mu A_\nu^a) A^{\mu b} A^{\nu c} - \frac{c^2}{4} \phi^2 \times f^{abc} f^{ade} A_\mu^b A_\nu^c A_\mu^d A^{\nu e}. \]  

(27)

There are also combinations of \( g = g_0 + c\phi \) with Chern–Simons couplings (like the 3D photon mass term in the mid-term exam and its non-abelian generalizations), but I don’t want to get into their details here.

* There might be some other allowed couplings in 3D, but never mind for now.

Finally, in \( d = 1 + 1 \) dimensions there is an infinite number of allowed \( \Delta \geq 0 \) couplings. Indeed, for \( d = 1 + 1 \) the bosonic fields have energy dimension \( E^0 \), so \( \Delta \) of a coupling does not depend on the number \( b \) of bosonic fields it involves but only on the numbers of derivatives and fermionic fields,

\[ \Delta = 2 - \delta - \frac{1}{2} f. \]  

(28)

Consequently, all scalar potentials \( V(\Phi) \) — including \( C_n \Phi^n \) terms for any \( n \), and even the non-polynomial potentials — have \( \Delta = +2 \), so any \( V(\Phi) \) is allowed in 2D. Likewise, all Yukawa-like couplings \( \Phi^n \bar{\Psi} \Psi \) have \( \Delta = +1 \), so we may have terms like \( y_{IJ}(\Phi) \times \bar{\Psi}^I \Psi^J \) for any functions \( y_{IJ}(\Phi) \).

At the \( \Delta = 0 \) level, we are allowed generic Riemannian metrics \( g_{ij}(\Phi) \) of the scalar field space, hence field-dependent kinetic terms

\[ \mathcal{L}_{\text{kin}} = \frac{1}{2} g_{ij}(\phi) \times \partial^\mu \phi^i \partial_\mu \phi^j, \]  

(29)
as well as a whole bunch of fermionic terms with arbitrary scalar-dependent coefficients,

\[ \mathcal{L}_\Psi \supset \frac{1}{4} g_{IJ}(\Phi) \times \bar{\Psi}^I \gamma^\mu \left( i \partial_\mu - i \partial^\mu \right) \Psi^J + \Gamma_{IJK}(\Phi) \times \partial_\mu \Phi^k \times \bar{\Psi}^I \gamma^\mu \Psi^J + \frac{1}{2} R_{IJKL}(\Phi) \times \bar{\Psi}^I \gamma^\mu \Psi^J \times \bar{\Psi}^K \gamma_\mu \Psi^L. \]  

(30)

In addition, there are gauge couplings with arbitrary scalar dependent \( g_{\text{gauge}}(\Phi) \), chiral couplings to Weyl or Majorana-Weyl fermions, etc., etc.