1. The EM vector potential $A_\mu(x)$ is subject to gauge transforms $A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)$. In spacetimes of higher dimensions $D > 4$ there are antisymmetric tensor fields subject to similar gauge transforms.

   Let's start with the 2-index antisymmetric tensor field $B_{\mu\nu}(x) \equiv -B_{\nu\mu}(x)$, where μ and ν are D-dimensional Lorentz indices running from 0 to $D-1$. To be precise, $B_{\mu\nu}(x)$ is the *tensor potential*, analogous to the electromagnetic vector potential $A_\mu(x)$. The analog of the EM tension fields $F_{\mu\nu}(x)$ is the 3–index tension tensor

$$H_{\lambda\mu\nu}(x) \;=\; \tfrac{1}{2}\partial_{[\lambda} B_{\mu\nu]} \;=\; \partial_\lambda B_{\mu\nu} \;+\; \partial_\mu B_{\nu\lambda} \;+\; \partial_\nu B_{\lambda\mu}\,. \tag{1}$$

   Note: this tensor is totally antisymmetric in all 3 indices.

   (a) Show that regardless of the Lagrangian, the H fields satisfy Jacobi identities

$$\tfrac{1}{6}\partial_{[\kappa} H_{\lambda\mu\nu]} \;\equiv\; \partial_\kappa H_{\lambda\mu\nu} \;-\; \partial_\lambda H_{\mu\nu\kappa} \;+\; \partial_\mu H_{\nu\kappa\lambda} \;-\; \partial_\nu H_{\kappa\lambda\mu} \;=\; 0. \tag{2}$$

   (b) The Lagrangian for the $B_{\mu\nu}(x)$ fields is

$$\mathcal{L}(B, \partial B) \;=\; \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \tag{3}$$

   where $H_{\lambda\mu\nu}$ are as in eq. (1). Treating the $B_{\mu\nu}(x)$ as $\frac{1}{2}D(D-1)$ independent fields, derive their equations of motion.

   Similar to the EM fields, the $B_{\mu\nu}$ fields are subject to *gauge transforms*

$$B'_{\mu\nu}(x) \;=\; B_{\mu\nu}(x) \;+\; \partial_\mu \Lambda_\nu(x) \;-\; \partial_\nu \Lambda_\mu(x) \tag{4}$$

   where $\Lambda_\mu(x)$ is an arbitrary vector field.

   (c) Show that the tension fields $H_{\lambda\mu\nu}(x)$ — and hence the Lagrangian (3) — are invariant under such gauge transforms.

In spacetimes of sufficiently high dimensions D, one may have similar tensor fields with more indices. Generally, the potentials form a p-index totally antisymmetric tensor $C_{\mu_1\mu_2\cdots\mu_p}(x)$, the tensions form a $p+1$ index tensor

$$G_{\mu_1\mu_2\cdots\mu_{p+1}}(x) \;=\; \frac{1}{p!}\,\partial_{[\mu_1}C_{\mu_2\cdots\mu_p\mu_{p+1}]}(x), \tag{5}$$

also totally antisymmetric in all its indices, and the Lagrangian is

$$\mathcal{L}(C,\partial C) \;=\; \frac{(-1)^p}{2(p+1)!}\,G_{\mu_1\mu_2\cdots\mu_{p+1}}G^{\mu_1\mu_2\cdots\mu_{p+1}}. \tag{6}$$

(d) Derive the Jacobi identities and the equations of motion for the G fields.

(e) Show that the tension fields $G_{\mu_1\mu_2\cdots\mu_{p+1}}(x)$ — and hence the Lagrangian (6) — are invariant under gauge transforms of the potentials $C_{\mu_1\mu_2\cdots\mu_p}(x)$ which act as

$$C'_{\mu_1\mu_2\cdots\mu_p}(x) \;=\; C_{\mu_1\mu_2\cdots\mu_p}(x) \;+\; \frac{1}{(p-1)!}\,\partial_{[\mu_1}\Lambda_{\mu_2\cdots\mu_p]}(x) \tag{7}$$

where $\Lambda_{\mu_2\cdots\mu_p}(x)$ is an arbitrary $(p-1)$-index tensor field (totally antisymmetric).

2. Next, consider the *massive* relativistic vector field $A^\mu(x)$ with Lagrangian density

$$\mathcal{L} \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \;+\; \tfrac{1}{2}m^2\,A_\mu A^\mu \;-\; A^\mu J_\mu \tag{8}$$

where $c=\hbar=1$, $F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$, and the current $J^\mu(x)$ is a fixed source for the $A^\mu(x)$ field. Note that because of the mass term, the Lagrangian (8) is *not* gauge invariant.

(a) Derive the Euler–Lagrange field equations for the massive vector field $A^\mu(x)$.

(b) Show that this field equation *does not require* current conservation; however, if the current happens to satisfy $\partial_\mu J^\mu = 0$, then the field $A^\mu(x)$ satisfies

$$\partial_\mu A^\mu \;=\; 0 \qquad\text{and}\qquad (\partial^2+m^2)A^\mu = J^\mu. \tag{9}$$

3. Finally, let's develop the Hamiltonian formalism for the massive vector field.

   (a) Our first step is to identify the canonically conjugate "momentum" fields. Show that $\partial\mathcal{L}/\partial\dot{\mathbf{A}} = -\mathbf{E}$ but $\partial\mathcal{L}/\partial\dot{A}_0 \equiv 0$.

   In other words, the canonically conjugate field to $\mathbf{A}(\mathbf{x})$ is $-\mathbf{E}(\mathbf{x})$ but the $A_0(\mathbf{x})$ does not have a canonical conjugate! Consequently,

$$H \;=\; \int\! d^3\mathbf{x}\left(-\dot{\mathbf{A}}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) \;-\; \mathcal{L}\right). \tag{10}$$

   (b) Show that in terms of the $\mathbf{A}$, $\mathbf{E}$, and A_0 fields, and their *space* derivatives,

$$H \;=\; \int\! d^3\mathbf{x}\left(\tfrac{1}{2}\mathbf{E}^2 \;+\; A_0\,(J_0 - \nabla\cdot\mathbf{E}) \;-\; \tfrac{1}{2}m^2A_0^2 \;+\; \tfrac{1}{2}\,(\nabla\times\mathbf{A})^2 \;+\; \tfrac{1}{2}m^2\mathbf{A}^2 \;-\; \mathbf{J}\cdot\mathbf{A}\right). \tag{11}$$

   Because the A_0 field does not have a canonical conjugate, the Hamiltonian formalism does not produce an equation for the time-dependence of this field. Instead, it gives us a time-independent equation relating the $A_0(\mathbf{x},t)$ to the values of other fields *at the same time* t:

$$\frac{\delta H}{\delta A_0(\mathbf{x})} \;\equiv\; \left.\frac{\partial\mathcal{H}}{\partial A_0}\right|_{\mathbf{x}} \;-\; \nabla\cdot\left.\frac{\partial\mathcal{H}}{\partial(\nabla A_0)}\right|_{\mathbf{x}} \;=\; 0. \tag{12}$$

   For the remaining fields $\mathbf{A}$ and $\mathbf{E}$ there are Hamiltonian equations of motion for their time derivatives, namely

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) \;&=\; -\left.\frac{\delta H}{\delta\mathbf{E}(\mathbf{x})}\right|_t \;\equiv\; -\left[\frac{\partial\mathcal{H}}{\partial\mathbf{E}} \;-\; \nabla_i\frac{\partial\mathcal{H}}{\partial(\nabla_i\mathbf{E})}\right]_{(\mathbf{x},t)}, \\ \frac{\partial}{\partial t}\mathbf{E}(\mathbf{x},t) \;&=\; +\left.\frac{\delta H}{\delta\mathbf{A}(\mathbf{x})}\right|_t \;\equiv\; +\left[\frac{\partial\mathcal{H}}{\partial\mathbf{A}} \;-\; \nabla_i\frac{\partial\mathcal{H}}{\partial(\nabla_i\mathbf{A})}\right]_{(\mathbf{x},t)}. \end{aligned} \tag{13}$$

   (c) Write down the explicit form of all these equations.

   (d) Verify that the equations you have just written down are equivalent to the relativistic Euler–Lagrange equations for the $A^\mu(x)$ you have obtained in problem 2.a.