1. In class I have introduced the (free) quantum scalar field $\hat{\varphi}(\mathbf{x},t)$, its canonically conjugate quantum field $\hat{\pi}(\mathbf{x},t)$, their equal-time commutation relations

$$\begin{aligned} \big[\hat{\varphi}(\mathbf{x},t),\hat{\varphi}(\mathbf{x}',\text{same }t)\big] &= 0, \\ \big[\hat{\pi}(\mathbf{x},t),\hat{\pi}(\mathbf{x}',\text{same }t)\big] &= 0, \\ \big[\hat{\varphi}(\mathbf{x},t),\hat{\pi}(\mathbf{x}',\text{same }t)\big] &= i\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \end{aligned} \tag{1}$$

and the Hamiltonian

$$\hat{H} = \int d^3\mathbf{x}\left(\tfrac{1}{2}\hat{\pi}^2(\mathbf{x}) + \tfrac{1}{2}\big(\nabla\hat{\varphi}(\mathbf{x})\big)^2 + \tfrac{1}{2}m^2\,\hat{\varphi}^2(\mathbf{x})\right). \tag{2}$$

I showed that $[\hat{\varphi}(\mathbf{x},t),\hat{H}] = i\hat{\pi}(\mathbf{x},t)$ and hence in the Heisenberg picture

$$\frac{\partial}{\partial t}\hat{\varphi}(\mathbf{x},t) = \hat{\pi}(\mathbf{x},t). \tag{3}$$

Your task is to show that

$$[\hat{\pi}(\mathbf{x},t),\hat{H}] = i\big(\nabla^2 - m^2\big)\hat{\varphi}(\mathbf{x},t) \tag{4}$$

and hence in the Heisenberg picture $\hat{\varphi}(\mathbf{x},t)$ satisfies the Klein–Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\hat{\varphi}(\mathbf{x},t) = 0. \tag{5}$$

2. Now let's quantize the massive vector field introduced in the previous homework (set#1, problems 2–3). In the classical Hamiltonian formalism, the space components $A^j(\mathbf{x})$ have canonical conjugates $-E^j(\mathbf{x})$, hence in the quantum theory, the quantum fields $\hat{A}^j(\mathbf{x},t)$ and $\hat{E}^k(\mathbf{x},t)$ satisfy equal-time commutation relations

$$\begin{aligned}
\left[\hat{A}^j(\mathbf{x},t), \hat{A}^k(\mathbf{x}', \text{same } t)\right] &= 0, \\
\left[\hat{E}^j(\mathbf{x},t), \hat{E}^k(\mathbf{x}', \text{same } t)\right] &= 0, \\
\left[\hat{A}^j(\mathbf{x},t), \hat{E}^k(\mathbf{x}', \text{same } t)\right] &= -i\delta^{jk}\delta^{(3)}(\mathbf{x}-\mathbf{x}').
\end{aligned} \tag{6}$$

The classical time component $A^0(\mathbf{x})$ does not have a canonical conjugate and hence no Hamilton equation for its time dependence. Instead, the $A^0(\mathbf{x})$ satisfies a time-independent *constraint* $m^2 A^0 = J^0 - \nabla \cdot \vec{E}$; in the quantum theory, this constraint is implemented as an *operatorial identity*

$$m^2 \hat{A}^0(\mathbf{x},t) \equiv \hat{J}^0(\mathbf{x},t) - \nabla \cdot \hat{\mathbf{E}}(\mathbf{x},t) \tag{7}$$

in the combined Hilbert space of the massive vector bosons and the charged particles giving rise to the $\hat{J}^0(\mathbf{x},t)$ and $\hat{\mathbf{J}}(\mathbf{x},t)$ operators. At equal times, the current operators $\hat{J}^0$ and $\hat{\mathbf{J}}$ commute with the vector fields $\hat{\mathbf{A}}$ and $\hat{\mathbf{E}}$, while the commutation relations of the $\hat{A}^0$ with the vector fields follow from eq. (7).

(a) Spell out the equal-times commutation relations of the $\hat{A}^0(\mathbf{x},t)$ field with the vector fields $\hat{\mathbf{A}}(\mathbf{x}', \text{same } t)$ and $\hat{\mathbf{E}}(\mathbf{x}', \text{same } t)$.

The Hamiltonian operator for the quantum vector fields follows from the classical Hamiltonian (eq. (11) from the previous homework):

$$\hat{H} = \int d^3\mathbf{x} \left(\tfrac{1}{2}\hat{\mathbf{E}}^2 + \hat{A}_0\left(\hat{J}_0 - \nabla\cdot\hat{\mathbf{E}}\right) - \tfrac{1}{2}m^2\hat{A}_0^2 + \tfrac{1}{2}\left(\nabla\times\hat{\mathbf{A}}\right)^2 + \tfrac{1}{2}m^2\hat{\mathbf{A}}^2 - \hat{\mathbf{J}}\cdot\hat{\mathbf{A}} \right). \tag{8}$$

Note that in the Heisenberg picture, all the quantum fields on the right hand side depend on time as well as $\mathbf{x}$, but the net Hamiltonian is time independent.

(b) Calculate the commutators of the vector fields $\hat{\mathbf{A}}(\mathbf{x},t)$ and $\hat{\mathbf{E}}(\mathbf{x},t)$ with the Hamiltonian (8), write down the Heisenberg equations for the quantum vector fields, and compare them to their classical counterparts from the previous homework.

3. An operator acting on identical bosons can be described in terms of N–particle wave functions (the *first-quantized* formalism) or in terms of creation and annihilation operators in the Fock space (the *second-quantized* formalism). This problem is about converting the operators from one formalism to another.

The key to this conversion are the single-particle wave functions $\phi_\alpha(\mathbf{x})$ of states $|\alpha\rangle$ and the *symmetrized* N-particle wave functions

$$\begin{aligned}\phi_{\alpha\beta\cdots\omega}(\mathbf{x}_1,\mathbf{x}_2\ldots,\mathbf{x}_N) &= \frac{1}{\sqrt{D}}\sum_{(\tilde\alpha,\tilde\beta,\ldots,\tilde\omega)}^{\substack{\text{distinct permutations}\\ \text{of }(\alpha,\beta,\ldots,\omega)}} \phi_{\tilde\alpha}(\mathbf{x}_1)\times\phi_{\tilde\beta}(\mathbf{x}_2)\times\cdots\times\phi_{\tilde\omega}(\mathbf{x}_N)\\ &= \frac{1}{T\sqrt{D}}\sum_{(\tilde\alpha,\tilde\beta,\ldots,\tilde\omega)}^{\substack{\text{all permutations}\\ \text{of }(\alpha,\beta,\ldots,\omega)}} \phi_{\tilde\alpha}(\mathbf{x}_1)\times\phi_{\tilde\beta}(\mathbf{x}_2)\times\cdots\times\phi_{\tilde\omega}(\mathbf{x}_N)\end{aligned} \tag{9}$$

of N-boson states $|\alpha,\beta,\ldots,\omega\rangle$. In eqs. (9), D is the number of *distinct* (*i.e.*, non-trivial) permutations of single-particle states $(\alpha,\beta,\ldots,\omega)$ and T is the number of trivial permutations. In terms of the occupation numbers n_γ

$$T = \prod_\gamma n_\gamma!\,, \qquad D = \frac{N!}{T}\,. \tag{10}$$

(a) Consider a generic N-particle quantum state $|N;\psi)\rangle$ with a totally symmetric wave-function $\psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)$. Show that the $(N+1)$–particle state $|N+1;\psi'\rangle = \hat a^\dagger_\alpha\,|N;\psi\rangle$ has wave function

$$\psi'(\mathbf{x}_1,\ldots,\mathbf{x}_{N+1}) = \frac{1}{\sqrt{N+1}}\sum_{i=1}^{N+1}\phi_\alpha(\mathbf{x}_i)\times\psi(\mathbf{x}_1,\ldots,\not{\mathbf{x}}_i,\ldots,\mathbf{x}_{N+1}). \tag{11}$$

Hint: First prove this for wave-functions of the form (9). Then use the fact that states $|\alpha_1,\ldots,\alpha_N\rangle$ form a complete basis of the N-boson Hilbert space.

(b) Show that the $(N-1)$–particle state $|N-1;\psi''\rangle = \hat a_\alpha\,|N;\psi\rangle$ has wave-function

$$\psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) = \sqrt{N}\int d^3\mathbf{x}_N\,\phi^*_\alpha(\mathbf{x}_N)\times\psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N). \tag{12}$$

Hint: the operator $\hat a_\alpha$ is the hermitian conjugate of $\hat a^\dagger_\alpha$, hence for any state $|N-1;\widetilde\psi\rangle$, $\langle N-1;\widetilde\psi|\,\hat a_\alpha\,|N;\psi\rangle = \langle N;\psi|\,\hat a^\dagger_\alpha\,|N-1;\widetilde\psi\rangle^*$.

Now consider the one-body operators, *i.e.* additive operators acting on one particle at a time. In the first-quantized formalism they act on N–particle states according to

$$\hat{A}_{\rm net}^{(1)} \;=\; \sum_{i=1}^{N} \hat{A}_1(i^{\rm th}\ \text{particle}) \tag{13}$$

where $\hat{A}_1$ is some kind of a one-particle operator (such as momentum $\hat{\mathbf{p}}$, or kinetic energy $\frac{1}{2m}\hat{\mathbf{p}}^2$, or potential $V(\hat{\mathbf{x}})$, *etc., etc.*). In the second-quantized formalism such operators become

$$\hat{A}_{\rm net}^{(2)} \;=\; \sum_{\alpha,\beta} \langle\alpha|\,\hat{A}_1\,|\beta\rangle\ \hat{a}_\alpha^\dagger \hat{a}_\beta\,. \tag{14}$$

(c) Verify that the two operators have the same matrix elements between any two N-boson states $|N;\psi\rangle$ and $|N;\widetilde{\psi}\rangle$, $\langle N;\widetilde{\psi}|\,\hat{A}_{\rm net}^{(1)}\,|N;\psi\rangle = \langle N;\widetilde{\psi}|\,\hat{A}_{\rm net}^{(2)}\,|N;\psi\rangle$.
Hint: use $\hat{A}_1 = \sum_{\alpha,\beta} |\alpha\rangle\,\langle\alpha|\,\hat{A}_1\,|\beta\rangle\,\langle\beta|$.

Finally, consider two-body operators, *i.e.* additive operators acting on two particles at a time. Given a two-particle operator $\hat{B}_2$ — such as $V(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2)$ — the *net* B operator acts in the first-quantized formalism according to

$$\hat{B}_{\rm net}^{(1)} \;=\; \tfrac{1}{2}\sum_{i\neq j} \hat{B}_2(i^{\rm th}\ \text{and}\ j^{\rm th}\ \text{particles}), \tag{15}$$

and in the second-quantized formalism according to

$$\hat{B}_{\rm net}^{(2)} \;=\; \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} \big(\langle\alpha|\otimes\langle\beta|\big)\hat{B}_2\big(|\gamma\rangle\otimes|\delta\rangle\big)\, \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta\,. \tag{16}$$

(d) Again, show these two operators have the same matrix elements between any two N-boson states, $\langle N;\widetilde{\psi}|\,\hat{A}_{\rm net}^{(1)}\,|N;\psi\rangle = \langle N;\widetilde{\psi}|\,\hat{A}_{\rm net}^{(2)}\,|N;\psi\rangle$ for any $\langle N;\widetilde{\psi}|$ and $|N;\psi\rangle$.

4. Finally, an exercise in bosonic commutation relations

$$[\hat{a}_\alpha, \hat{a}_\beta] \;=\; 0, \qquad [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] \;=\; 0, \qquad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] \;=\; \delta_{\alpha\beta}\,. \tag{17}$$

(a) Calculate the commutators $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma]$, $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta]$ and $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta]$.

(b) Consider three one-particle operators $\hat{A}_1$, $\hat{B}_1$, and $\hat{C}_1$. Let us define the corresponding second-quantized operators $\hat{A}^{(2)}_{\text{net}}$, $\hat{B}^{(2)}_{\text{net}}$, and $\hat{C}^{(2)}_{\text{net}}$ according to eq. (14).

Show that if $\hat{C}_1 = [\hat{A}_1, \hat{B}_1]$ then $\hat{C}^{(2)}_{\text{net}} = \left[\hat{A}^{(2)}_{\text{net}}, \hat{B}^{(2)}_{\text{net}}\right]$.

(c) Next, calculate the commutator $[\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta, \hat{a}^\dagger_\mu \hat{a}_\nu]$.

(d) Now let $\hat{A}_1$ be a one-particle operator, let $\hat{B}_2$ and $\hat{C}_2$ be two-body operators, and let $\hat{A}^{(2)}_{\text{net}}$, $\hat{B}^{(2)}_{\text{net}}$, and $\hat{C}^{(2)}_{\text{net}}$ be the corresponding second-quantized operators according to eqs. (14) and (16).

Show that if $\hat{C}_2 = \left[\left(\hat{A}_1(1^{\text{st}}) + \hat{A}_1(2^{\text{nd}})\right), \hat{B}_2\right]$ then $\hat{C}^{(2)}_{\text{net}} = \left[\hat{A}^{(2)}_{\text{net}}, \hat{B}^{(2)}_{\text{net}}\right]$.

(e) Finally, show that for any analytic function $f(\hat{a}^\dagger)$ of the creation operators or for any function $g(\hat{a})$ of the annihilation operators,

$$\begin{aligned}
[\hat{a}_\alpha, f(\hat{a}^\dagger)] \;&=\; \frac{\partial f(\hat{a}^\dagger)}{\partial \hat{a}^\dagger_\alpha}\,, \qquad [\hat{a}^\dagger_\alpha, g(\hat{a})] \;=\; -\frac{\partial g(\hat{a})}{\partial \hat{a}_\alpha}\,, \\
\exp\left(\sum\nolimits_\alpha c_\alpha \hat{a}_\alpha\right) f(\hat{a}^\dagger) \exp\left(-\sum\nolimits_\alpha c_\alpha \hat{a}_\alpha\right) \;&=\; f(\text{each } \hat{a}^\dagger_\alpha \to \hat{a}^\dagger_\alpha + c_\alpha), \\
\exp\left(\sum\nolimits_\alpha c_\alpha \hat{a}^\dagger_\alpha\right) g(\hat{a}) \exp\left(-\sum\nolimits_\alpha c_\alpha \hat{a}^\dagger_\alpha\right) \;&=\; g(\text{each } \hat{a}_\alpha \to \hat{a}_\alpha - c_\alpha).
\end{aligned} \tag{18}$$

Hint: two of these four equations are hermitian conjugates of the other two.