1. Quantum mechanics of a fixed number of *relativistic* particles may be a useful approximation for some systems, but it's *inconsistent* as a complete theory. Among other problems, it allows superluminal propagation of particles, which is inconsistent with relativistic causality. Indeed, consider a single free relativistic spinless particle with Hamiltonian

$$\hat{H} \;=\; +\sqrt{M^2+\hat{\mathbf{P}}^2} \tag{1}$$

(in the $c=\hbar=1$ units). In the coordinate picture, this Hamiltonian is a horrible integro-differential operator, but that's only a technical problem. The real problem concerns the time evolution kernel

$$U(\mathbf{x}-\mathbf{y};t) \;=\; \langle\mathbf{x},t|\mathbf{y},t_0=0\rangle^{\text{Heisenberg}}_{\text{picture}} \;=\; \langle\mathbf{x}|\exp(-it\hat{H})\,|\mathbf{y}\rangle^{\text{Schroedinger}}_{\text{picture}}\,. \tag{2}$$

(a) Show that

$$U(\mathbf{x}-\mathbf{y};t) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\exp\bigl(i(\mathbf{x}-\mathbf{y})\mathbf{k}-it\omega(k)\bigr) \;=\; \frac{-i}{4\pi^2\,r}\int\limits_{-\infty}^{+\infty}\!dk\,k\,\exp\bigl(irk-it\omega(k)\bigr), \tag{3}$$

where $r=|\mathbf{x}-\mathbf{y}|$ and $\omega(k)=+\sqrt{M^2+k^2}$.

(b) Take the limit $t\to\infty$, $r\to\infty$, with fixed ratio r/t; let's stay inside the future light cone, so $(r/t)<1$. Show that in this limit, the evolution kernel becomes

$$U(\mathbf{x}-\mathbf{y};t) \;\approx\; \frac{(-iM)^{3/2}}{4\pi^{3/2}}\,\frac{t}{(t^2-r^2)^{5/4}}\times\exp(-iM\sqrt{t^2-r^2}). \tag{4}$$

Hint: Use the saddle point method to evaluate the integral (3). If you are not familiar with this method — or any other method for approximating integrals of the form $\int dk\,f(x)\times\exp\bigl(Ag(x)\bigr)$ in the limit $A\to\infty$ — then read *my notes on the saddle-point method.* (I wrote those notes for a QM class, hence the Airy function example. For this problem, you don't need that example, just the general method.)

(c) Finally, take a similar limit but go outside the light cone, thus fixed $(r/t) > 1$ while $r, t \to \infty$. Show that in this limit, the kernel becomes

$$U(\mathbf{x}-\mathbf{y};t) \;\approx\; \frac{iM^{3/2}}{4\pi^{3/2}}\,\frac{t}{(r^2-t^2)^{5/4}} \times \exp(-M\sqrt{r^2-t^2}). \tag{5}$$

This formula shows that the kernel diminishes exponentially outside the light cone, *but it does not vanish!* Thus, given a particle localized at point $\mathbf{y}$ at the time $t_0 = 0$, after time $t > 0$, its wave function is *mostly* limited to the future light cone $r < t$, *but there is an exponential tail outside the light cone.* In other words, the probability of superluminal motion is exponentially small but non-zero.

Obviously, such superluminal propagation cannot be allowed in a consistently relativistic theory. And that's why relativistic quantum mechanics of a single particle is inconsistent. Likewise, relativistic quantum mechanics of any fixed number of particles does not work, except as an approximation.

In the quantum field theory, this paradox is resolved by allowing for creation and annihilation of particles. Quantum field operators acting at points x and y outside each others' light cones can either create a particle at x and then annihilate it at y, or else annihilate it at y and then create it at x. I will show in class that the two effects *precisely* cancel each other, so altogether there is no propagation outside the light cone. That's how relativistic QFT is perfectly causal while the relativistic QM is not.

2. Similar to the scalar field I have discussed in class, any free relativistic field $\widehat{\Psi}_{\aleph}(x)$ — where $\aleph$ stands for a vector, tensor, or spinor index or milti-index — can be expanded into creation and annihilation operators:

$$\widehat{\Psi}_{\aleph}(x) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\sum_{\lambda}\Bigl(e^{-ikx}U_{\aleph}(\mathbf{k},\lambda)\,\hat{a}_{\mathbf{k},\lambda} \;+\; e^{+ikx}V_{\aleph}(\mathbf{k},\lambda)\,\hat{a}^{\dagger}_{\mathbf{k},\lambda}\Bigr)^{k^0=+\omega_{\mathbf{k}}} \tag{6}$$

for a real (hermitian) quantum field, or

$$\begin{aligned}
\widehat{\Psi}_{\aleph}(x) \;&=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\sum_{\lambda}\Bigl(e^{-ikx}U_{\aleph}(\mathbf{k},\lambda)\,\hat{a}_{\mathbf{k},\lambda} \;+\; e^{+ikx}V_{\aleph}(\mathbf{k},\lambda)\,\hat{b}^{\dagger}_{\mathbf{k},\lambda}\Bigr)^{k^0=+\omega_{\mathbf{k}}}, \\
\widehat{\Psi}^{\dagger}_{\aleph}(x) \;&=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\sum_{\lambda}\Bigl(e^{-ikx}V^{*}_{\aleph}(\mathbf{k},\lambda)\,\hat{b}_{\mathbf{k},\lambda} \;+\; e^{+ikx}U^{*}_{\aleph}(\mathbf{k},\lambda)\,\hat{a}^{\dagger}_{\mathbf{k},\lambda}\Bigr)^{k^0=+\omega_{\mathbf{k}}}
\end{aligned} \tag{7}$$

for a complex field and its hermitian conjugate. In all cases, $kx \equiv k_\mu x^\mu = \omega_k t - \mathbf{k}\cdot\mathbf{x}$ for $\omega_k = +\sqrt{\mathbf{k}^2 + m^2}$,

$$e^{-ikx} \times U_\aleph(\mathbf{k},\lambda) \quad \text{and} \quad e^{+ikx} \times V_\aleph(\mathbf{k},\lambda) \tag{8}$$

are plane-wave solutions of the classical field equations; the *polarizations* — *i.e.*, independent solutions for the same k^μ — are labeled by λ.

In particular, a free massive vector field $A^\mu(x)$ satisfies $(\partial^2 + m^2)A^\mu = 0$ and $\partial_\mu A^\mu = 0$ (see homework set#1, problem 2), hence the plane-wave solutions have form

$$A^\mu(x) \;=\; e^{-ikx}\times\mathcal{E}^\mu_{\mathbf{k},\lambda} \quad \text{or} \quad A^\mu(x) \;=\; e^{+ikx}\times\mathcal{E}^{\mu*}_{\mathbf{k},\lambda} \quad \text{for} \quad k^\mu = (+\omega_k,\mathbf{k}) \quad \text{and} \quad k_\mu\mathcal{E}^\mu_{\mathbf{k},\lambda} = 0. \tag{9}$$

For each $\mathbf{k}$ there are 3 independent choices of $\mathcal{E}^\mu$ vectors, hence 3 polarizations $\lambda = 1,2,3$ (or $\lambda = -1,0,+1$ in the helicity basis). For $\mathbf{k} = 0$ the $\mathcal{E}^\mu_{\mathbf{0},\lambda}$ are 3 purely-space vectors of unit length and $\perp$ to each other; for other $\mathbf{k}$, we Lorentz-boost these 3 vectors into the moving particle's frame.

(a) Show that such boost gives not only $k_\mu\mathcal{E}^\mu_{\mathbf{k},\lambda} = 0$ for all 3 polarizations but also

$$g_{\mu\nu}\mathcal{E}^\mu_{\mathbf{k},\lambda}\mathcal{E}^{\nu*}_{\mathbf{k},\lambda'} \;=\; -\delta_{\lambda,\lambda'} \quad \text{and} \quad \sum_\lambda \mathcal{E}^\mu_{\mathbf{k},\lambda}\mathcal{E}^{\nu*}_{\mathbf{k},\lambda} \;=\; -g^{\mu\nu} \;+\; \frac{k^\mu k^\nu}{m^2}\,. \tag{10}$$

Note: the $\mathcal{E}^\mu_{\mathbf{k},\lambda}$ could be real or complex, dependent on a polarization (*e.g.*, planar or circular), so it's important to distinguish between the $\mathcal{E}^\mu_{\mathbf{k},\lambda}$ and the $\mathcal{E}^{\mu*}_{\mathbf{k},\lambda}$.

Applying the general rule (6) to the free massive real vector fields gives

$$\hat{A}^\mu(x) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_\lambda\left(e^{-ikx}\mathcal{E}^\mu_{\mathbf{k},\lambda}\,\hat{a}_{\mathbf{k},\lambda} \;+\; e^{+ikx}\mathcal{E}^{\mu*}_{\mathbf{k},\lambda}\,\hat{a}^\dagger_{\mathbf{k},\lambda}\right)^{k^0=+\omega_\mathbf{k}}\,. \tag{11}$$

(b) Show that

$$\hat{F}^{\mu\nu}(x) \;=\; \partial^{[\mu}A^{\nu]}(x) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_\lambda\left(e^{-ikx}\mathcal{E}^{\mu\nu}_{\mathbf{k},\lambda}\,\hat{a}_{\mathbf{k},\lambda} \;+\; e^{+ikx}\mathcal{E}^{\mu\nu*}_{\mathbf{k},\lambda}\,\hat{a}^\dagger_{\mathbf{k},\lambda}\right)^{k^0=+\omega_\mathbf{k}} \tag{12}$$

where $\mathcal{E}^{\mu\nu}_{\mathbf{k},\lambda} = -ik^\mu\mathcal{E}^\nu_{\mathbf{k},\lambda} + ik^\nu\mathcal{E}^\mu_{\mathbf{k},\lambda}$.

(c) Specialize eqs. (11) and (12) to the $\hat{A}^0(x)$ and $\hat{\mathbf{E}}(x)$ and show that they are consistent with the operatorial identity $m^2\hat{A}^0 + \nabla\cdot\hat{\mathbf{E}} \equiv 0$ for the free massive vector field. (*cf.* eq. (7) from the last homework).

The creation / annihilation operators $\hat{a}^\dagger_{\mathbf{k},\lambda}$ and $\hat{a}_{\mathbf{k},\lambda}$ satisfy the (relativistically normalized) bosonic commutation relations

$$\left[\hat{a}_{\mathbf{k},\lambda},\hat{a}_{\mathbf{k}',\lambda'}\right] = 0, \quad \left[\hat{a}^\dagger_{\mathbf{k},\lambda},\hat{a}^\dagger_{\mathbf{k}',\lambda'}\right] = 0, \quad \left[\hat{a}_{\mathbf{k},\lambda},\hat{a}^\dagger_{\mathbf{k}',\lambda'}\right] = \delta_{\lambda\lambda'}\times 2\omega_k(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{13}$$

(d) Show that these relations lead to the same *equal-time* commutation relations for the vector fields $\hat{\mathbf{A}}$ and $\hat{\mathbf{E}}$ as we had in eq. (6) of the last homework.

The rest of this problem is about the Hamiltonian of the free massive vector fields and its expansion into products of creation and annihilation operators.

(e) First, take $\hat{H}$ from eq. (8) of the last homework and show that in the absence of the current $\hat{J}^\mu(x)$

$$\hat{H} = \int d^3\mathbf{x}\left(\tfrac{1}{4}\,\delta_{\alpha\mu}\delta_{\beta\nu}\hat{F}^{\alpha\beta}\hat{F}^{\mu\nu} + \tfrac{1}{2}m^2\,\delta_{\alpha\mu}\hat{A}^\alpha\hat{A}^\mu\right) \tag{14}$$

Note: the $\delta_{\alpha\mu}$ and $\delta_{\beta\nu}$ here are not typos for $g_{\alpha\mu}$ and $g_{\beta\nu}$; the integrand here is *not* Lorentz invariant.

(f) Before expanding the quantum fields in eq. (14) into creation and annihilation operators, show that the polarization vectors satisfy

$$\begin{aligned}\tfrac{1}{4}\,\delta_{\alpha\mu}\delta_{\beta\nu}\mathcal{E}^{\alpha\beta}_{\mathbf{k},\lambda}\mathcal{E}^{\mu\nu *}_{\mathbf{k},\lambda'} + \tfrac{1}{2}m^2\,\delta_{\alpha\mu}\mathcal{E}^{\alpha}_{\mathbf{k},\lambda}\mathcal{E}^{\mu *}_{\mathbf{k},\lambda'} &= \omega^2_{\mathbf{k}}\times\delta_{\lambda,\lambda'}\,,\\ \tfrac{1}{4}\,\delta_{\alpha\mu}\delta_{\beta\nu}\mathcal{E}^{\alpha\beta}_{\mathbf{k},\lambda}\mathcal{E}^{\mu\nu}_{-\mathbf{k},\lambda'} + \tfrac{1}{2}m^2\,\delta_{\alpha\mu}\mathcal{E}^{\alpha}_{\mathbf{k},\lambda}\mathcal{E}^{\mu}_{-\mathbf{k},\lambda'} &= 0.\end{aligned} \tag{15}$$

Hint: use $k_\mu\mathcal{E}^\mu_{\mathbf{k},\lambda} = 0$ and eqs. (10). Note that $\delta_{\alpha\mu}k^\alpha\mathcal{E}^\mu_{\mathbf{k},\lambda}$ is different from $k_\mu\mathcal{E}^\mu_{\mathbf{k},\lambda}$ and that the 3-momentum $\mathbf{k}' = -\mathbf{k}$ comes with energy $k'^0 = +k^0 = +\omega_{\mathbf{k}}$.

(g) Now expand the $\hat{A}^\mu(x)$ and $\hat{F}^{\mu\nu}$ fields in eq. (14) into creation and annihilation operators according to eqs. (11) and (12), integrate over $\mathbf{x}$, use eqs. (15) to simplify the sums over polarizations, all to show that

$$\hat{H} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \sum_\lambda \omega_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} + \text{const.} \tag{16}$$

The constant term here is an infinite c-number which results from commuting $\hat{a}^\dagger_{\mathbf{k},\lambda}$ through $\hat{a}_{\mathbf{k},\lambda}$ for the same $\mathbf{k}$ and λ. Despite this infinity, this term commutes with all the quantum fields so it plays no role in their dynamics. Usually, this term is ignored or defined away (by redefining $\hat{H}$ in terms of normal-ordered products of the quantum fields), and that's what we are going to do in this homework.

(h) Finally, verify that the time-dependence of the quantum fields (11) and (12) agrees with the Heisenberg equations for the Hamiltonian (16),

$$i\partial_0 \hat{A}^\mu(x) = \left[\hat{A}^\mu(x), \hat{H}\right], \quad i\partial_0 \hat{F}^{\mu\nu}(x) = \left[\hat{F}^{\mu\nu}(x), \hat{H}\right]. \tag{17}$$

3. The last problem is about the Feynman propagator of the massive vector field.

(a) Calculate the "vacuum sandwich" of two vector fields and show that

$$\begin{aligned} \langle 0| \hat{A}^\mu(x)\hat{A}^\nu(y) |0\rangle &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \left[\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right) e^{-ik(x-y)}\right]^{k^0=+\omega_{\mathbf{k}}} \\ &= \left(-g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right) D(x-y). \end{aligned} \tag{18}$$

(b) And now, the Feynman propagator: Show that

$$\begin{aligned} G_F^{\mu\nu} \equiv \langle 0| \mathbf{T}^* \hat{A}^\mu(x)\hat{A}^\nu(y) |0\rangle &= \left(-g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right) G_F^{\text{scalar}}(x-y) \\ &= \int \frac{d^4\mathbf{k}}{(2\pi)^4} \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right) \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i0} \end{aligned} \tag{19}$$

where

$$\mathbf{T}^* \hat{A}^\mu(x)\hat{A}^\nu(y) = \mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y) + \frac{i}{m^2}\,\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y), \tag{20}$$

is the *modified* time-ordered product of the vector fields. The purpose of this modification* is to absorb the $\delta^{(4)}(x-y)$ stemming from the $\partial_0\partial_0 G_F(x-y)$.

Finally, the classical action for the free massive vector field may be written in the form

$$S \;=\; \tfrac{1}{2}\int d^4x\, A^\mu(x)\,\mathcal{D}_{\mu\nu}A^\nu(x) \qquad \text{where } \mathcal{D}_{\mu\nu} \;=\; g_{\mu\nu}(\partial^2+m^2) \;-\; \partial_\mu\partial_\nu\,. \tag{21}$$

(c) Verify this formula and show that the Feynman propagator (19) is a Green's function of the same $\mathcal{D}_{\mu\nu}$ differential operator as in eq. (21), namely

$$\mathcal{D}_{\mu\nu}[\text{acting on } x]G_F^{\nu\lambda}(x-y) \;=\; +i\delta_\mu^\lambda\delta^{(4)}(x-y). \tag{22}$$

$\star$ See *Quantum Field Theory* by Claude Itzykson and Jean–Bernard Zuber.