1. First, an exercise in Dirac matrices γ^μ. Please do not assume any specific form of these 4×4 matrices, just use the anti-commutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{1}$$

In class, I have defined the spin matrices

$$S^{\mu\nu} = -S^{\nu\mu} \stackrel{\rm def}{=} \tfrac{i}{4}\left[\gamma^\mu,\gamma^\nu\right] \tag{2}$$

and showed that

$$\left[S^{\mu\nu},\gamma^\lambda\right] = ig^{\nu\lambda}\gamma^\mu - ig^{\mu\lambda}\gamma^\nu. \tag{3}$$

(a) Show that the spin matrices $S^{\mu\nu}$ have commutation relations of the Lorentz generators,

$$\left[S^{\kappa\lambda},S^{\mu\nu}\right] = ig^{\lambda\mu}S^{\kappa\nu} - ig^{\lambda\nu}S^{\kappa\mu} - ig^{\kappa\mu}S^{\lambda\nu} + ig^{\kappa\nu}S^{\lambda\mu}. \tag{4}$$

A continuous Lorentz transform obtains from integrating infinite sequences of infinitesimal transforms $X'^\mu = X^\mu + \epsilon\,\Theta^\mu_{\ \nu}X^\nu$ for antisymmetric $\Theta_{\mu\nu} = -\Theta_{\nu\mu}$ (when both indices are down or both up); the finite transform is $X'^\mu = L^\mu_{\ \nu}X^\nu$ where

$$L = \exp(\Theta), \quad i.e., \quad L^\mu_{\ \nu} = \delta^\mu_\nu + \Theta^\mu_{\ \nu} + \tfrac12\Theta^\mu_{\ \lambda}\Theta^\lambda_{\ \nu} + \tfrac16\Theta^\mu_{\ \kappa}\Theta^\kappa_{\ \lambda}\Theta^\lambda_{\ \nu} + \cdots. \tag{5}$$

(b) Let L be a Lorentz transform of the form (5), and let $M_D(L) = \exp\bigl(-\frac{i}{2}\Theta_{\alpha\beta}S^{\alpha\beta}\bigr)$. Show that $M_D^{-1}(L)\gamma^\mu M_D(L) = L^\mu_{\ \nu}\gamma^\nu$.
Hint: use Hadamard Lemma $e^ABe^{-A} = B+[A,B]+\frac12[A,[A,B]]+\frac16[A,[A,[A,B]]]+\cdots$.

Next, a little more algebra:

(c) Calculate $\{\gamma^\rho,\gamma^\lambda\gamma^\mu\gamma^\nu\}$, $[\gamma^\rho,\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu]$ and $[S^{\rho\sigma},\gamma^\lambda\gamma^\mu\gamma^\nu]$.

(d) Show that $\gamma^\alpha\gamma_\alpha = 4$, $\gamma^\alpha\gamma^\nu\gamma_\alpha = -2\gamma^\nu$, $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = 4g^{\mu\nu}$, and $\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha = -2\gamma^\nu\gamma^\mu\gamma^\lambda$.
Hint: use $\gamma^\alpha\gamma^\nu = 2g^{\nu\alpha} - \gamma^\nu\gamma^\alpha$ repeatedly.

2. Now consider the $\gamma^5 \stackrel{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$ matrix.

   (a) Show that γ^5 anticommutes with each of the γ^μ matrices — $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ — and commutes with all the spin matrices $S^{\mu\nu}$.

   (b) Show that γ^5 is hermitian and that $(\gamma^5)^2 = 1$.

   (c) Show that $\gamma^5 = (i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ and $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = -24i\epsilon^{\kappa\lambda\mu\nu}\,\gamma^5$.

   (d) Show that $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = -6i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5$.

   (e) Show that any 4×4 matrix Γ is a unique linear combination of the following 16 matrices: 1, γ^μ, $\frac{1}{2}\gamma^{[\mu}\gamma^{\nu]}$, $\gamma^5\gamma^\mu$, and γ^5.

   * My conventions here are: $\epsilon^{0123} = -1$, $\epsilon_{0123} = +1$, $\gamma^{[\mu}\gamma^{\nu]} = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$, $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\lambda\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\lambda$, *etc.*

   Now consider Dirac matrices in spacetime dimensions $d \neq 4$. Such matrices always satisfy the Clifford algebra (1), but their sizes depend on d.

   Let $\Gamma = i^n\gamma^0\gamma^1\cdots\gamma^{d-1}$ be the generalization of the γ^5 to d dimensions; the pre-factor $i^n = \pm i$ or ± 1 is chosen such that $\Gamma = \Gamma^\dagger$ and $\Gamma^2 = +1$.

   (f) *For even* d, Γ anticommutes with all the γ^μ. Prove this, and use this fact to show that there are 2^d independent products of the γ^μ matrices, and consequently the matrices should be $2^{d/2}\times 2^{d/2}$.

   (g) *For odd* d, Γ commutes with all the Γ^μ — prove this. Consequently, one can set $\Gamma = +1$ or $\Gamma = -1$; the two choices lead to in-equivalent sets of the γ^μ.

   Classify the independent products of the γ^μ for odd d and show that their net number is 2^{d-1}; consequently, the matrices should be $2^{(d-1)/2}\times 2^{(d-1)/2}$.

3. Let's go back to $d = 3+1$. Since all the spin matrices $S^{\mu\nu}$ commute with the γ^5, all the $M_D(L) = \exp\bigl(-\frac{i}{2}\Theta_{\alpha\beta}S^{\alpha\beta}\bigr)$ matrices are block-diagonal in the eigenbasis of the γ^5. In the Weyl convention for the γ matrices,

$$M_D(L) \;=\; \begin{pmatrix} M_L(L) & 0 \\ 0 & M_R(L) \end{pmatrix} \tag{6}$$

   where all blocks are 2×2. This makes the Dirac spinor a *reducible* representation of the continuous Lorentz group $SO^+(3,1)$.

Write down the explicit $S^{\mu\nu}$ matrices in the Weyl convention, then use them to show that the $M_L(L)$ block in eq. (6) is precisely the $SL(2,\mathbf{C})$ matrix M from problem 4 of the last homework while the M_R block is the $\overline{M} = \left(M^\dagger\right)^{-1} = \sigma_2 M^* \sigma_2$ matrix from the same problem. In particular, for a pure rotation through angle φ around axis $\mathbf{n}$

$$M_L \;=\; M_R \;=\; \exp(-\tfrac{i}{2}\varphi\,\mathbf{n}\cdot\boldsymbol{\sigma}). \tag{7}$$

while for a pure boost of *rapidity* r in the direction $\mathbf{n}$, $M_L = \exp(-\frac{r}{2}\,\mathbf{n}\cdot\boldsymbol{\sigma})$ but $M_R = \exp(+\frac{r}{2}\,\mathbf{n}\cdot\boldsymbol{\sigma})$; in terms of the boost's β and γ parameters,

$$M_L \;=\; \sqrt{\gamma}\times\sqrt{1\,-\,\beta\,\mathbf{n}\cdot\boldsymbol{\sigma}}\,,\qquad M_R \;=\; \sqrt{\gamma}\times\sqrt{1\,+\,\beta\,\mathbf{n}\cdot\boldsymbol{\sigma}}\,. \tag{8}$$

4. Finally, consider the plane-wave solutions $e^{-ipx}u(p,s)$ and $e^{+ipx}v(p,x)$ of the Dirac equation. The 4–component spinors $u(p,s)$ and $v(p,s)$ satisfy

$$(\not{p}-m)u(p,s) \;=\; 0,\quad (\not{p}+m)v(p,s) \;=\; 0,\quad u^\dagger(p,s)u(p,s') \;=\; v^\dagger(p,s)v(p,s') \;=\; 2E\delta_{s,s'}\,. \tag{9}$$

Let's writing down explicit formulae for these spinors in the Weyl basis for the γ^μ matrices.

(a) Show that for $\mathbf{p}=0$,

$$u(\mathbf{p}=\mathbf{0},s) \;=\; \begin{pmatrix} \sqrt{m}\,\xi_s \\ \sqrt{m}\,\xi_s \end{pmatrix} \tag{10}$$

where ξ_s is a two-component $SO(3)$ spinor encoding the electron's spin state. The ξ_s are normalized to $\xi_s^\dagger \xi_{s'} = \delta_{s,s'}$.

(b) For other momenta, $u(p,s) = M(\text{boost})u(\mathbf{p}=0,s)$ for the boost that turns $(m,\vec{0})$ to p^μ. Use eqs. (8) to show that

$$u(p,s) \;=\; \begin{pmatrix} \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix}. \tag{11}$$

(c) Use similar arguments to show that

$$v(p,s) \;=\; \begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix} \tag{12}$$

where η_s are two-component $SO(3)$ spinors normalized to $\eta_s^\dagger \eta_{s'} = \delta_{s,s'}$.

Physically, the η_s should have opposite spins from ξ_s — the holes in the Dirac sea have opposite spins (as well as p^μ) from the missing negative-energy particles. Mathematically, this requires $\eta_s^\dagger \mathbf{S} \eta_s = -\xi_s^\dagger \mathbf{S} \xi_s$; we may solve this condition by letting $\eta_s = \sigma_2 \xi_s^* = \pm i \xi_{-s}$.

(d) Check that this is a solution, then show that it leads to $v(p,s) = \gamma^2 u^*(p,s)$.

(e) Show that for ultra-relativistic electrons or positrons of definite helicity $\lambda = \pm\frac{1}{2}$, the Dirac plane waves become *chiral* — *i.e.*, dominated by one of the two irreducible components $\mathbf{2}$ or $\bar{\mathbf{2}}$ of the Dirac spinor $\mathbf{2} \oplus \bar{\mathbf{2}}$ while the other component becomes negligible. (The $\mathbf{2}$ component is the left-handed Weyl spinor while the $\bar{\mathbf{2}}$ component is the right-handed Weyl spinor. I shall discuss them later in class.) Specifically,

$$\begin{aligned} u(p,-\tfrac{1}{2}) &\approx \sqrt{2E} \begin{pmatrix} \xi_L \\ 0 \end{pmatrix}, & u(p,+\tfrac{1}{2}) &\approx \sqrt{2E} \begin{pmatrix} 0 \\ \xi_R \end{pmatrix}, \\ v(p,-\tfrac{1}{2}) &\approx -\sqrt{2E} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix}, & v(p,+\tfrac{1}{2}) &\approx \sqrt{2E} \begin{pmatrix} \eta_R \\ 0 \end{pmatrix}. \end{aligned} \tag{13}$$

Note that for electrons the left/right chirality is same as the helicity, but for positrons the chirality is opposite from the helicity.

Finally, let's establish some basis-independent properties of the Dirac spinors $u(p,s)$ and $v(p,s)$ — although you may use the Weyl basis to verify them.

(f) Show that

$$\bar{u}(p,s) u(p,s') = +2m\delta_{s,s'}, \quad \bar{v}(p,s) v(p,s') = -2m\delta_{s,s'}; \tag{14}$$

note that the normalization here is different from eq. (9) for the $v^\dagger u$ and $v^\dagger v$.

(g) There are only two independent $SO(3)$ spinors, hence $\sum_s \xi_s \xi_s^\dagger = \sum_s \eta_s^\dagger \eta_s = \mathbf{1}_{2\times 2}$. Use this fact to show that

$$\sum_{s=1,2} u_\alpha(p,s) \bar{u}_\beta(p,s) = (\not{p}+m)_{\alpha\beta} \quad \text{and} \quad \sum_{s=1,2} v_\alpha(p,s) \bar{v}_\beta(p,s) = (\not{p}-m)_{\alpha\beta}. \tag{15}$$