This homework is about discrete symmetries of Dirac fermions, the charge conjugation **C** and the parity (reflection of space) **P**.

1. Let's start with the charge conjugation **C** which exchanges particles with antiparticles, for example the electrons e^- with positrons e^+,

$$\widehat{\mathbf{C}}\left|e^-(\mathbf{p},s)\right\rangle \;=\; \left|e^+(\mathbf{p},s)\right\rangle, \quad \widehat{\mathbf{C}}\left|e^+(\mathbf{p},s)\right\rangle \;=\; \left|e^-(\mathbf{p},s)\right\rangle. \tag{1}$$

Note that the operator $\widehat{\mathbf{C}}$ is unitary and squares to one (repeating the exchange brings us back to the original particles), hence $\widehat{\mathbf{C}}^\dagger = \widehat{\mathbf{C}}^{-1} = \widehat{\mathbf{C}}$. For other species of Dirac fermions, the $F \leftrightarrow \overline{F}$ exchange may be accompanied by an overall sign which depends on a species F,

$$\widehat{\mathbf{C}}\left|F(\mathbf{p},s)\right\rangle \;=\; \pm\left|\overline{F}(\mathbf{p},s)\right\rangle, \quad \widehat{\mathbf{C}}\left|\overline{F}(\mathbf{p},s)\right\rangle \;=\; \pm\left|F(\mathbf{p},s)\right\rangle, \tag{2}$$

same sign in both formulae.

(a) In the fermionic Fock space, the $\widehat{\mathbf{C}}$ operator act on multi-particle states by turning each particle into an antiparticle and vice verse according to eqs. (2). Show that this action implies

$$\widehat{\mathbf{C}}\,\hat{a}^\dagger_{\mathbf{p},s}\widehat{\mathbf{C}} \;=\; \pm\hat{b}^\dagger_{\mathbf{p},s}\,, \quad \widehat{\mathbf{C}}\,\hat{b}^\dagger_{\mathbf{p},s}\widehat{\mathbf{C}} \;=\; \pm\hat{a}^\dagger_{\mathbf{p},s}\,, \quad \widehat{\mathbf{C}}\,\hat{a}_{\mathbf{p},s}\widehat{\mathbf{C}} \;=\; \pm\hat{b}_{\mathbf{p},s}\,, \quad \widehat{\mathbf{C}}\,\hat{b}_{\mathbf{p},s}\widehat{\mathbf{C}} \;=\; \pm\hat{a}_{\mathbf{p},s}\,. \tag{3}$$

(b) The quantum Dirac fields $\widehat{\Psi}(x)$ and $\widehat{\overline{\Psi}}(x)$ are linear combinations of creation and annihilation operators. Use eqs. (3) and the plane-wave relations $v(p,s) = \gamma^2 u^*(p,s)$ and $u(p,s) = \gamma^2 v^*(p,s)$ from the last homework to show that

$$\widehat{\mathbf{C}}\widehat{\Psi}(x)\widehat{\mathbf{C}} \;=\; \pm\gamma^2\widehat{\Psi}^*(x) \quad \text{and} \quad \widehat{\mathbf{C}}\widehat{\overline{\Psi}}(x)\widehat{\mathbf{C}} \;=\; \pm\widehat{\overline{\Psi}}^*(x)\gamma^2 \tag{4}$$

where $*$ stands for an hermitian conjugation of the component fields but without transposing a column vector (of 4 Dirac components) into a row vector or vice verse, thus $\widehat{\Psi}^* = \left(\widehat{\Psi}^\dagger\right)^\top$, $\left(\widehat{\Psi}^\dagger\right)^* = \widehat{\Psi}^\top$, and $\widehat{\overline{\Psi}}^* = \widehat{\Psi}^\top(\gamma^0)^* = \widehat{\Psi}^\top\gamma^0$.

(c) Show that the Dirac equation transforms covariantly under the charge conjugation (4). Hint: prove and use $\gamma^\mu\gamma^2 = -\gamma^2(\gamma^\mu)^*$ for all γ^μ in the Weyl basis.

(d) Show that that the *classical* Dirac Lagrangian is invariant under the charge conjugation (up to a total spacetime derivative). Note that in the classical limit the Dirac fields *anticommute* with each other, $\Psi^*_\alpha\Psi_\beta = -\Psi_\beta\Psi^*_\alpha$. Also, similar to the hermitian conjugation of quantum fields, the complex conjugation of fermionic fields reverses their order: $(F_1F_2)^* = F_2^*F_1^* = -F_1^*F_2^*$.

2. Now consider the *parity* $\mathbf{P}$, the im-proper Lorentz symmetry that reflects the space but not the time, $(\mathbf{x}, t) \to (-\mathbf{x}, +t)$. This symmetry acts on Dirac spinor fields according to

$$\widehat{\Psi}'(-\mathbf{x}, +t) \;=\; \pm\gamma^0\widehat{\Psi}(+\mathbf{x}, +t) \tag{5}$$

where the overall $\pm$ sign is *intrinsic parity* of the fermion species.

(a) Verify that the Dirac equation transforms covariantly under (5) and that the Dirac Lagrangian is invariant (apart from $\mathcal{L}(\mathbf{x}, t) \to \mathcal{L}(-\mathbf{x}, t)$).

In the Fock space, eq. (5) becomes

$$\widehat{\mathbf{P}}\widehat{\Psi}(\mathbf{x}, t)\widehat{\mathbf{P}} \;=\; \pm\gamma^0\widehat{\Psi}(-\mathbf{x}, t) \tag{6}$$

for some unitary operator $\widehat{\mathbf{P}}$ that squares to one. Let's find how this operator acts on the particles and their states.

(b) First, look up the plane-wave solutions $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ in the last homework and show that $u(-\mathbf{p}, s) = +\gamma^0 u(\mathbf{p}, s)$ while $v(-\mathbf{p}, s) = -\gamma^0 v(\mathbf{p}, s)$.

(c) Now show that eq. (6) implies

$$\begin{aligned}
\widehat{\mathbf{P}}\,\hat{a}_{\mathbf{p},s}\,\widehat{\mathbf{P}} \;&=\; \pm\hat{a}_{-\mathbf{p},+s}\,, \quad & \widehat{\mathbf{P}}\,\hat{a}^\dagger_{\mathbf{p},s}\,\widehat{\mathbf{P}} \;&=\; \pm\hat{a}^\dagger_{-\mathbf{p},+s}\,,\\
\widehat{\mathbf{P}}\,\hat{b}_{\mathbf{p},s}\,\widehat{\mathbf{P}} \;&=\; \mp\hat{b}_{-\mathbf{p},+s}\,, \quad & \widehat{\mathbf{P}}\,\hat{b}^\dagger_{\mathbf{p},s}\,\widehat{\mathbf{P}} \;&=\; \mp\hat{b}^\dagger_{-\mathbf{p},+s}\,,
\end{aligned} \tag{7}$$

and hence

$$\widehat{\mathbf{P}}\,|F(\mathbf{p}, s)\rangle \;=\; \pm\,|F(-\mathbf{p}, +s)\rangle \quad \text{and} \quad \widehat{\mathbf{P}}\,\big|\overline{F}(\mathbf{p}, s)\big\rangle \;=\; \mp\,\big|\overline{F}(-\mathbf{p}, +s)\big\rangle\,. \tag{8}$$

Note that the fermion and the antifermion have opposite intrinsic parities!

3. Some electrically neutral particles carry other kinds of changes (forex, the baryon number) that distinguish them from their antiparticles. But other particles — such as the photon or the π^0 meson — have no charges at all and act as their own antiparticles. The charge conjugation symmetry turns such particles n into themselves,

$$\widehat{\mathbf{C}}\,|n(\mathbf{p},s)\rangle \;=\; \pm\,|n(\mathbf{p},s)\rangle\,, \tag{9}$$

where the overall $\pm$ sign is called the *C-parity* or *charge-parity* of the particle in question. This C-parity — as well as the P-parity under space reflections — limit the allowed decay channels of unstable particles via strong and EM interactions which respect both $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{P}}$ symmetries.

Consider a bound state of a charged Dirac fermion F and the corresponding antifermion, for example a $q\bar{q}$ meson or a positronium "atom" (a hydrogen-atom-like bound state of e^- and e^+). Suppose this bound state has a definite orbital angular momentum L and a definite net spin S.

Show that the C-parity and the P-parity of this bound state are

$$C \;=\; (-1)^{L+S}, \quad P \;=\; (-1)^{L+1}. \tag{10}$$

Hint: In the Fock space of fermions and antifermions, the bound state with zero net momentum is

$$|B(\mathbf{p}_{\text{tot}}=0)\rangle \;=\; \int\!\frac{d^3\mathbf{p}_{\text{red}}}{(2\pi)^3}\sum_{s_1,s_2}\psi(\mathbf{p}_{\text{red}},s_1,s_2)\times\hat{a}^\dagger(+\mathbf{p}_{\text{red}},s_1)\,\hat{b}^\dagger(-\mathbf{p}_{\text{red}},s_2)\,|0\rangle \tag{11}$$

for some wave-function ψ of the reduced momentum and the two spins. The orbital angular momentum controls the symmetry of ψ with respect to $\mathbf{p}_{\text{red}}\to-\mathbf{p}_{\text{red}}$ while the net spin S controls the symmetry of ψ under $s_1\leftrightarrow s_2$.

Also, explain why the annihilation rate of the ground 1S state of the positronium "atom" depends on the net spin: the $S=0$ state decays much faster than the $S=1$ state. Note: since the EM fields couple linearly to the electric charges and currents (which are reversed by $\widehat{\mathbf{C}}$), each photon has $C=-1$.

4. Finally, consider bilinear products of a Dirac field $\Psi(x)$ and its conjugate $\overline{\Psi}(x)$. Generally, such products have form $\overline{\Psi}\Gamma\Psi$ where Γ is one of 16 matrices discussed in the previous homework; altogether, we have

$$S = \overline{\Psi}\Psi, \quad V^\mu = \overline{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \overline{\Psi}\tfrac{i}{2}\gamma^{[\mu}\gamma^{\nu]}\Psi, \quad A^\mu = \overline{\Psi}\gamma^5\gamma^\mu\Psi, \quad \text{and} \quad P = \overline{\Psi}i\gamma^5\Psi. \tag{12}$$

(a) Show that all the bilinears (12) are Hermitian.
Hint: First, show that $\left(\overline{\Psi}\Gamma\Psi\right)^\dagger = \overline{\Psi}\Gamma\Psi$.
Note: despite the Fermi statistics, $\left(\Psi^\dagger_\alpha\Psi_\beta\right)^\dagger = +\Psi^\dagger_\beta\Psi_\alpha$.

(b) Show that under *continuous* Lorentz symmetries, the S and the P transform as scalars, the V^μ and the A^μ as vectors, and the $T^{\mu\nu}$ as an antisymmetric tensor.

(c) Find the transformation rules of the bilinears (12) under parity and show that while S is a true scalar and V is a true (polar) vector, P is a pseudoscalar and A is an axial vector.

Next, consider the charge-conjugation properties of the Dirac bilinears. To avoid the operator-ordering problems, take the classical limit where $\Psi(x)$ and $\Psi^\dagger(x)$ *anticommute* with each other, $\Psi_\alpha\Psi^\dagger_\beta = -\Psi^\dagger_\beta\Psi_\alpha$.

(d) Show that **C** turns $\overline{\Psi}\Gamma\Psi$ into $\overline{\Psi}\Gamma^c\Psi$ where $\Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2$.

(e) Calculate Γ^c for all 16 independent matrices Γ and find out which Dirac bilinears are C–even and which are C–odd.