1. Consider an $O(N)$ symmetric Lagrangian for N interacting real scalar fields,

$$\mathcal{L} \;=\; \frac{1}{2}\sum_{a=1}^{N}\bigl(\partial_\mu\Phi_a\bigr)^2 \;-\; \frac{m^2}{2}\sum_{a=1}^{N}\Phi_a^2 \;-\; \frac{\lambda}{24}\left(\sum_{a=1}^{N}\Phi_a^2\right)^2 . \tag{1}$$

By the Noether theorem, the continuous $SO(N)$ subgroup of (N) symmetry gives rise to $\frac{1}{2}N(N-1)$ conserved currents

$$J^\mu_{ab}(x) \;=\; -J^\mu_{ba}(x) = \;\Phi_a(x)\,\partial^\mu\Phi_b(x) - \;\Phi_b(x)\,\partial^\mu\Phi_a(x). \tag{2}$$

In the quantum field theory, these currents become operators

$$\hat{J}^\mu_{ab}(x) \;=\; -\hat{J}^\mu_{ba}(x) = \;\hat{\Phi}_a(x)\,\partial^\mu\hat{\Phi}_b(x) \;-\; \hat{\Phi}_b(x)\,\partial^\mu\hat{\Phi}_a(x), \tag{3}$$

or in the Schrödinger picture,

$$\hat{\mathbf{J}}_{ab}(\mathbf{x}) \;=\; -\hat{\Phi}_a(\mathbf{x})\nabla\hat{\Phi}_b(\mathbf{x}) \;+\; \hat{\Phi}_b(\mathbf{x})\nabla\hat{\Phi}_a(\mathbf{x}), \quad \hat{J}^0_{ab}(\mathbf{x}) \;=\; \hat{\Phi}_a(\mathbf{x})\hat{\Pi}_b(\mathbf{x}) \;-\; \hat{\Phi}_b(\mathbf{x})\hat{\Pi}_a(\mathbf{x}). \tag{4}$$

This problem is about the net charge operators

$$\hat{Q}_{ab} \;=\; -\hat{Q}_{ba} \;=\int\! d^3\mathbf{x}\,\hat{J}^0(\mathbf{x}) \;=\int\! d^3\mathbf{x}\,\Bigl(\hat{\Phi}_a(\mathbf{x})\hat{\Pi}_b(\mathbf{x}) \;-\; \hat{\Phi}_b(\mathbf{x})\hat{\Pi}_a(\mathbf{x})\Bigr). \tag{5}$$

(a) Use equal-time commutation relations of the $\hat{\Phi}$ and $\hat{\Pi}$ fields to show that

$$\Bigl[\hat{Q}_{ab},\Phi_c(\mathbf{x})\Bigr] \;=\; i\delta_{bc}\hat{\Phi}_a(\mathbf{x}) - i\delta_{ac}\hat{\Phi}_b(\mathbf{x}) \quad\text{and}\quad \Bigl[\hat{Q}_{ab},\Pi_c(\mathbf{x})\Bigr] \;=\; i\delta_{bc}\hat{\Pi}_a(\mathbf{x}) - i\delta_{ac}\hat{\Pi}_b(\mathbf{x}). \tag{6}$$

(b) Write down the Hamiltonian operator for the interacting fields and verify that it commutes with all the charges. In the Heisenberg picture, this makes all the charge operators $\hat{Q}_{ab}$ time independent.

(c) Verify that the $\hat{Q}_{ab}$ obey commutation relations of the $SO(N)$ generators,

$$\left[\hat{Q}_{ab}, \hat{Q}_{cd}\right] \;=\; -i\delta_{[c[b[}\hat{Q}_{a]d]} \;\equiv\; -i\delta_{bc}\hat{Q}_{ad} \;+\; i\delta_{ac}\hat{Q}_{bd} \;+\; i\delta_{bd}\hat{Q}_{ac} \;-\; i\delta_{ad}\hat{Q}_{bc}\,. \tag{7}$$

(d) In the Schrödinger picture $\hat{\Phi}_a(\mathbf{x})$ and $\hat{\Pi}_a(\mathbf{x})$ can be expanded into creation and annihilation operators as if they were free fields. Show that in terms of creation and annihilation operators, the charges (5) become

$$\hat{Q}_{ab} \;=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\left(-i\hat{a}^\dagger_{\mathbf{p},a}\hat{a}_{\mathbf{p},b} \;+\; i\hat{a}^\dagger_{\mathbf{p},b}\hat{a}_{\mathbf{p},b}\right). \tag{8}$$

Now consider a finite symmetry $R_{ab} \in SO(N)$; in general, $R = \exp(\Theta)$ for some real antisymmetric $N \times N$ matrix $\Theta_{ab} = -\Theta_{ba}$. In the Fock space of the scalar field theory, this symmetry is represented by the unitary operator

$$\widehat{\mathcal{D}}(R) \;=\; \exp\left(-\tfrac{i}{2}\Theta_{ab}\hat{Q}_{ab}\right) \tag{9}$$

where inside the exponent there is implicit sum over a and b.

(e) Verify that this operators acts on single-particle states $|\mathbf{p}, a\rangle$ as appropriate 'rotation' of the a index,

$$\widehat{\mathcal{D}}(R)\,|\mathbf{p}, a\rangle \;=\; R_{ab}\,|\mathbf{p}, b\rangle\,. \tag{10}$$

(f) The charges (8) are additive one-body-at-a-time operators. Use this fact to derive the action of the $\widehat{\mathcal{D}}(R)$ operator on multi-particle states.

Finally, for $N = 2$ the $SO(2)$ symmetry is the phase symmetry of one complex field $\Phi = (\Phi_1 + i\Phi_2)/\sqrt{2}$ and its conjugate $\Phi^* = (\Phi_1 - i\Phi_2)/\sqrt{2}$. In the Fock space, they give rise to particles and anti-particles of opposite charges.

(g) show that for $N = 2$

$$\hat{Q}_{21} \;=\; -\hat{Q}_{12} \;=\; \hat{N}_{\text{particles}} \;-\; \hat{N}_{\text{antiparticles}} \;=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \;-\; \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}}\right). \tag{11}$$

2. The rest of this homework is about *classical* fields and their stress-energy tensors.

   As I explained in class, applying Noether theorem to translational symmetries $x^\mu \to x^\mu + c^\mu$ of a classical field theory produces 4 conserved currents packaged into a 2-index stress-energy tensor

$$T_N^{\mu\nu} = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi^a \;-\; g^{\mu\nu}\,\mathcal{L}. \tag{12}$$

   In this formula, the ϕ_a are generic fields and a stands for all of their indices, including Lorentz vector, tensor, or spinor indices, internal symmetry indices (if any), and even labels distinguishing field species unrelated by any symmetries.

   Unfortunately, for non-scalar fields the Noether stress-energy tensor (12) is not symmetric, $T_N^{\mu\nu} \neq T_N^{\nu\mu}$. To make a symmetric stress-energy tensor — which is required for conserved currents for Lorentz symmetries

$$\mathcal{M}^{\lambda,\mu\nu} = -\mathcal{M}^{\lambda,\nu\mu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + \mathcal{M}^{\lambda,\mu\nu}_{\rm spin}, \quad \partial_\lambda\mathcal{M}^{\lambda,\mu\nu} = 0 \tag{13}$$

   as well as coupling to General Relativity — one adds a total divergence to the Noether stress-energy tensor,

$$T^{\mu\nu} = T^{\mu\nu}_{\rm Noether} + \partial_\lambda\mathcal{K}^{\lambda\mu,\nu}, \tag{14}$$

   where $\mathcal{K}^{\lambda\mu,\nu} = -\mathcal{K}^{\mu\lambda,\nu}$ is some 3–index Lorentz tensor antisymmetric in its first two indices. It's specific form as a function of the fields and their derivatives is whatever it takes to make the stress-energy tensor symmetric. $T^{\mu\nu} = T^{\nu\mu}$.

   (a) Show that regardless of the specific form of $\mathcal{K}^{\lambda\mu,\nu}(\phi,\partial\phi)$,

$$\partial_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu}_{\rm Noether} = (\text{hopefully}) = 0, \tag{15}$$

$$P^\mu_{\rm net} \equiv \int d^3\mathbf{x}\, T^{0\mu} = \int d^3\mathbf{x}\, T^{0\mu}_{\rm Noether}\,. \tag{16}$$

   Note: assume that the fields — and hence $\mathcal{K}$ — go to zero fast enough for $\mathbf{x}\to\infty$.

As an example of $\mathcal{K}$–corrected stress-energy tensor (14), consider the free electromagnetic fields $A^\mu(x)$ with Lagrangian

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) \;=\; -\tfrac{1}{4}\, F_{\mu\nu}F^{\mu\nu} \quad \text{where} \quad F_{\mu\nu} \;\equiv\; \partial_\mu A_\nu \;-\; \partial_\nu A_\mu \,. \tag{17}$$

(b) Write down the Noether stress-energy tensor $T_N^{\mu\nu}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.

(c) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T_{\text{EM}}^{\mu\nu} \;=\; -F^{\mu\lambda}F^{\nu}_{\ \lambda} \;+\; \tfrac{1}{4}\, g^{\mu\nu}\, F_{\kappa\lambda}F^{\kappa\lambda}. \tag{18}$$

Show that this expression indeed has form (14) for some $\mathcal{K}^{\lambda\mu,\nu}$.

(d) Write down the components of the stress-energy tensor (18) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density and pressure.

Now consider the electromagnetic fields coupled to the electric current J^μ of some charged "matter" fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate P^μ_{EM} and P^μ_{mat}. Consequently, we should have

$$\partial_\mu T_{\text{net}}^{\mu\nu} \;=\; 0 \qquad \text{for} \quad T_{\text{net}}^{\mu\nu} \;=\; T_{\text{EM}}^{\mu\nu} \;+\; T_{\text{mat}}^{\mu\nu} \tag{19}$$

but generally $\partial_\mu T_{\text{EM}}^{\mu\nu} \neq 0$ and $\partial_\mu T_{\text{mat}}^{\mu\nu} \neq 0$.

(e) Use Maxwell's equations to show that

$$\partial_\mu T_{\text{EM}}^{\mu\nu} \;=\; -F^{\nu\lambda}J_\lambda \tag{20}$$

and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_λ according to

$$\partial_\mu T_{\text{mat}}^{\mu\nu} \;=\; +F^{\nu\lambda}J_\lambda. \tag{21}$$

3. As a simplest example of charged matter, consider a complex scalar field $\Psi(x)$ of electric charge $q \neq 0$. The net Lagrangian of the Φ, Φ^* and A^μ fields is

$$\mathcal{L}_{\rm net} \;=\; D^\mu\Phi^* D_\mu\Phi \;-\; m^2\Phi^*\Phi \;-\; \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{22}$$

where

$$D_\mu\Phi \;=\; (\partial_\mu + iqA_\mu)\Phi \quad\text{and}\quad D_\mu\Phi^* \;=\; (\partial_\mu - iqA_\mu)\Phi^* \tag{23}$$

are the *covariant* derivatives. Through these derivatives, the Lagrangian (22) depend on the EM potentials A^μ and not just the EM tensions $F^{\mu\nu}$, which leads to a non-trivial electric current

$$J^\mu \;\stackrel{\rm def}{=}\; -\frac{\partial\mathcal{L}}{\partial A_\mu}\,. \tag{24}$$

(a) Write down the equation of motion for all fields in a covariant form. Also, write down the electric current (24) in a manifestly gauge-invariant form and verify its conservation, $\partial_\mu J^\mu = 0$ (as long as the scalar fields satisfy their equations of motion).

(b) Write down the Noether stress-energy tensor for the whole field system and show that

$$T^{\mu\nu}_{\rm net} \;\equiv\; T^{\mu\nu}_{\rm EM} \;+\; T^{\mu\nu}_{\rm mat} \;=\; T^{\mu\nu}_{\rm Noether} \;+\; \partial_\lambda\mathcal{K}^{\lambda\mu,\nu}, \tag{25}$$

where $T^{\mu\nu}_{\rm EM}$ is exactly as in eq. (18) for the free EM fields, the $\mathcal{K}^{\lambda\mu,\nu}$ tensor is also exactly as in the previous problem for the free EM, while

$$T^{\mu\nu}_{\rm mat} \;=\; D^\mu\Phi^*\,D^\nu\Phi \;+\; D^\nu\Phi^*\,D^\mu\Phi \;-\; g^{\mu\nu}\big(D_\lambda\Phi^*\,D^\lambda\Phi \;-\; m^2\Phi^*\,\Phi\big). \tag{26}$$

Note: In the presence of an electric current J^μ, the $\partial_\lambda\mathcal{K}^{\lambda\mu\nu}$ correction to the electromagnetic stress-energy tensor contains an extra $J^\mu A^\nu$ term. This term is important for obtaining a gauge-invariant stress-energy tensor (26) for the scalar field.

(c) Use the scalar fields' equations of motion and the non-commutativity of covariant derivatives

$$[D_\mu, D_\nu]\Phi \;=\; iqF_{\mu\nu}\Phi, \qquad [D_\mu, D_\nu]\Phi^* \;=\; -iqF_{\mu\nu}\Phi^* \tag{27}$$

to verify eq. (21) for the charged scalar fields and hence conservation of the net stress-energy tensor (25).