1. In class, I have focused on the *fundamental multiplet* of the local $SU(N)$ symmetry, *i.e.*, a set of N fields (complex scalars or Dirac fermions) which transform as a complex N–vector,

$$\Psi'(x) \;=\; U(x)\Psi(x) \quad i.e. \quad \Psi'_i(x) \;=\sum_j U_i^{\ j}(x)\Psi_j(x), \quad i,j=1,2,\ldots,N \tag{1}$$

where $U(x)$ is an x–dependent unitary $N\times N$ matrix, $\det U(x)\equiv 1$. Now consider N^2-1 real fields $\Phi^a(x)$ forming an *adjoint multiplet*: In matrix form

$$\Phi(x) \;=\; \sum_a \Phi^a(x)\times\frac{\lambda^a}{2} \tag{2}$$

is a traceless hermitian $N\times N$ matrix which transforms under the local $SU(N)$ symmetry as

$$\Phi'(x) \;=\; U(x)\Phi(x)U^\dagger(x). \tag{3}$$

Note that this transformation law preserves the $\Phi^\dagger=\Phi$ and $\mathrm{tr}(\Phi)=0$ conditions.

In class, I have argued (using covariant derivatives) that the tension fields $\mathcal{F}^a_{\mu\nu}(x)$ themselves transform according to eq. (3). In other words, the $\mathcal{F}^a_{\mu\nu}(x)$ form an adjoint multiplet of the $SU(N)$ symmetry group.

(a) Verify the $\mathcal{F}'_{\mu\nu}(x)=U(x)\mathcal{F}_{\mu\nu}(x)U^\dagger(x)$ transformation law directly from the definition $\mathcal{F}_{\mu\nu}\stackrel{\rm def}{=}\partial_\mu\mathcal{A}_\nu-\partial_\mu\mathcal{A}_\nu+i[\mathcal{A}_\mu,\mathcal{A}_\nu]^\star$ and the non-abelian gauge transform of the $\mathcal{A}_\mu$ fields.

Now consider an adjoint multiplet of some fields $\Phi^a(x)$ — the $\mathcal{F}^a_{\mu\nu}(x)$, or some real scalar fields, or Majorana fermions, whatever. The covariant derivatives D_μ act on such

$\star$ In my notations A_μ and $F_{\mu\nu}$ are canonically normalized fields while $\mathcal{A}_\mu=gA_\mu$ and $\mathcal{F}_{\mu\nu}=gF_{\mu\nu}$ are normalized by the symmetry action.

multiplet as

$$D_\mu\Phi(x) \;=\; \partial_\mu\Phi(x) \;+\; i[\mathcal{A}_\mu(x),\Phi(x)] \;\equiv\; \partial_\mu\Phi(x) \;+\; i\mathcal{A}_\mu(x)\Phi(x) \;-\; i\Phi(x)\mathcal{A}_\mu(x) \tag{4}$$

(b) Verify that these derivatives are indeed covariant — the $D_\mu\Phi(x)$ transforms under the local $SU(N)$ symmetry exactly like the $\Phi(x)$ itself.

(c) Show that $[D_\mu, D_\nu]\Phi(x) = i[\mathcal{F}_{\mu\nu}(x),\Phi(x)]$.

(d) Now let's go back to the tension fields $\mathcal{F}_{\mu\nu}(x)$ and verify the non-abelian Bianchi identity $D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0$.

(e) Show that for an infinitesimal variation of the non-abelian gauge field $A_\nu(x) \to A_\nu(x) + \delta A_\nu(x)$, the tension varies according to $\delta F_{\mu\nu}(x) = D_\mu\delta A_\nu(x) - D_\nu\delta A_\mu(x)$.

Now consider the non-abelian gauge theory coupled to some currents J^a_μ,

$$\begin{aligned}\mathcal{L} \;&=\; -\frac{1}{4g^2}\sum_a \mathcal{F}^a_{\mu\nu}\mathcal{F}^{a\mu\nu} \;-\; \sum_a \mathcal{A}^{a\mu}J^a_\mu \\ &=\; -\frac{1}{2g^2}\,\mathrm{tr}\,(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) \;-\; 2\,\mathrm{tr}(\mathcal{A}^\mu J_\mu).\end{aligned} \tag{5}$$

The currents J^a_μ follow from $\mathcal{A}^\mu$ appearing in covariant derivatives of some scalar or fermionic fields, but let's keep them un-specific for a moment.

(f) Write down classical equations of motions for the gauge fields.

(g) Show that consistency of those equations require the currents to be *covariantly conserved,*

$$D_\mu J^\mu \;=\; \partial_\mu J^\mu \;+\; i[\mathcal{A}_\mu, J^\mu] \;=\; 0, \tag{6}$$

or in components, $\partial_\mu J^{a\mu} - f^{abc}\mathcal{A}^b_\mu J^{c\mu} = 0$.

Note: a covariantly conserved current does *not* lead to a conserved charge, $(d/dt)\int d^3\mathbf{x}\,J^{a0}(\mathbf{x},t) \neq 0$!

2. Now consider a general Lie group G with generators $\hat{T}^a$ obeying commutation relations $[\hat{T}^a, \hat{T}^b] = if^{abc}\hat{T}^c$. Under an infinitesimal gauge symmetry

$$\mathcal{G}(x) \;=\; 1 \;+\; i\Lambda^a(x)\hat{T}^a \;+\; \cdots, \quad \text{infinitesimal } \Lambda^a(x), \tag{7}$$

the gauge fields $\mathcal{A}^a_\mu(x)$ transform as

$$\mathcal{A}^a_\mu(x) \;\to\; \mathcal{A}^a_\mu(x) \;-\; D_\mu\Lambda^a(x) \;=\; \mathcal{A}^a_\mu(x) \;-\; \partial_\mu\Lambda^a(x) \;-\; f^{abc}\Lambda^b(x)\mathcal{A}^c_\mu(x). \tag{8}$$

In any multiplet (m) of G, the generators are represented by matrices $(T^a_{(m)})_\alpha^{\;\beta}$ satisfying similar commutation relations, $[T^a_{(m)}, T^b_{(m)}] = if^{abc}T^c_{(m)}$. Fields $\Phi_\alpha(x)$ (fermionic, scalar, or whatever) belonging to some multiplet (m) transform under infinitesimal gauge transforms (7) as

$$\Phi_\alpha(x) \;\to\; \Phi_\alpha(x) \;+\; i\Lambda^a(x)(T^a_{(m)})_\alpha^{\;\beta}\Phi_\beta(x). \tag{9}$$

The covariant derivatives D_μ act on these fields as

$$D_\mu\Phi_\alpha(x) \;=\; \partial_\mu\Phi_\alpha(x) \;+\; i\mathcal{A}^a_\mu(x)(T^a_{(m)})_\alpha^{\;\beta}\Phi_\beta(x). \tag{10}$$

(a) Verify covariance of these derivatives under infinitesimal gauge transforms (7).

⋆ For extra challenge, prove covariance of the derivatives (10) under finite gauge transforms. This question is only for students familiar with basic theory of Lie groups.

Now consider Dirac fields $\Psi_\alpha(x)$ in some multiplet (m) of a simple non-abelian gauge group G. The combined Lagrangian for the fermion and gauge fields is

$$\mathcal{L} \;=\; -\frac{1}{4g^2}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{a\mu\nu} \;+\; \overline{\Psi}^\alpha(i\gamma^\mu D_\mu - m)\Psi_\alpha\,. \tag{11}$$

(b) Write down the currents $J^a_\mu = -\partial\mathcal{L}_\Psi/\partial\mathcal{A}^\mu_a$ of this theory and show that they form an adjoint multiplet of G. That is, show that under infinitesimal gauge symmetries (7), the currents transform into each other as

$$J^a_\mu(x) \;\to\; J^a_\mu(x) \;-\; f^{abc}\Lambda^b(x)J^c_\mu(x). \tag{12}$$

Note: in the adjoint multiplet $(T^a_{\text{adj}})^{bc} = -if^{abc}$; the commutations relations for the T^a_{adj} follow from the Jacobi identity for the Lie algebra generators.

- ⋆ For extra challenge, show that for finite gauge transforms the currents transform according to the adjoint representation of the group G. Again, this challenge is only to students familiar with Lie groups.

(c) Finally, verify that the currents $J^a_\mu(x)$ are covariantly conserved, $D_\mu J^{a\mu} = 0$, provided the fermionic fields $\Psi_\alpha(x)$ and $\overline{\Psi}^\alpha(x)$ satisfy their equations of motion.

3. A Dirac spinor field $\Psi(x)$ (together with its conjugate $\overline{\Psi}(x)$) is equivalent to two left-handed Weyl spinor fields $\chi(x)$ and $\tilde{\chi}(x)$ (together with their right-hand conjugates $\sigma_2\chi^*(x)$ and $\sigma^2\tilde{\chi}^*(x)$). In the Weyl basis (where γ^5 is diagonal)

$$\Psi(x) \;=\; \begin{pmatrix} \chi(x) \\ -\sigma_2\tilde{\chi}^*(x) \end{pmatrix}, \qquad \overline{\Psi}(x) \;=\; \Big(-\tilde{\chi}^\top(x)\sigma_2\,,\, \chi^\dagger(x)\Big). \tag{13}$$

(a) Show that up to a total derivative

$$\mathcal{L}_{\text{Dirac}} \;\equiv\; \overline{\Psi}(i\,\partial\!\!\!/ - m)\Psi \;=\; i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi \;+\; i\tilde{\chi}^\dagger\bar{\sigma}^\mu\partial_\mu\tilde{\chi} \;+\; m\chi^\top\sigma_2\,\tilde{\chi} \;+\; m\chi^\dagger\sigma_2\,\tilde{\chi}^*. \tag{14}$$

Hint: $\sigma_2\sigma^\mu\sigma_2 = (\bar{\sigma}^\mu)^* = (\bar{\sigma}^\mu)^\top$.
Note the $\chi \leftrightarrow \tilde{\chi}$ symmetry of the Lagrangian (14): In the last two terms, the σ^2 matrix is antisymmetric but the fields are fermionic, hence $\chi^\top\sigma_2\tilde{\chi} = -\tilde{\chi}^\top\sigma_2^\top\chi = +\tilde{\chi}^\top\sigma_2\chi$ and likewise $\chi^\dagger\sigma_2\tilde{\chi}^* = +\tilde{\chi}^\dagger\sigma_2\chi^*$.

(b) What happen to the LH spinor fields $\chi(x)$ and $\tilde{\chi}(x)$ and to the Lagrangian (14) under an axial symmetry?

(c) Work out how parity $\mathbf{P} : (\mathbf{x}, t) \to (-\mathbf{x}, +t)$, charge conjugation $\mathbf{C}$, and the combined $\mathbf{CP}$ symmetry act on the Weyl spinor fields $\chi(x)$ and $\tilde{\chi}(x)$.

Now consider N left-handed Weyl spinor fields $\chi_j(x)$ with free Lagrangian

$$\mathcal{L} \;=\; \sum_j i\chi^{j\dagger}\bar{\sigma}^\mu\partial_\mu\chi_j \;+\; \tfrac{1}{2}\sum_{j,k} M^{jk}\chi_j^\top\sigma_2\,\chi_k \;+\; \tfrac{1}{2}\sum_{j,k} M^*_{jk}\chi^{j\dagger}\sigma_2\,\chi^{k*}. \tag{15}$$

The mass matrix M^{jk} here must be symmetric, $M^{jk} = M^{kj}$, but it may be complex rather than real.

(d) Show that the Weyl equations for the χ_j fields lead to Klein–Gordon equations with mass^2 matrix $M^*M = M^\dagger M$, hence the physical fermion masses^2 are eigenvalues of the $M^\dagger M$.
Hints: $\sigma_2(\sigma^\mu)^\top\sigma_2 = \bar{\sigma}^\mu$; $\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2g^{\mu\nu}$.

Now consider the combined **CP** symmetry of the Weyl fermions. In the simplest case, the symmetry acts similarly on all the spinors,

$$\mathbf{CP} : \chi_j(\mathbf{x}, t) \ \to = \ \pm i \times \sigma_2\chi_j^*(-\mathbf{x}, +t), \quad \text{same } \pm i \ \forall j. \tag{16}$$

The overall factor here is $\pm i$ rather than ± 1 because for fermions $(\mathbf{CP})^2 = -1$. Note that having the same overall factor $\pm i$ for all spinors is different from what we had for the χ and the $\tilde{\chi}$ in part (c).

(e) Show that the free Lagrangian (15) is invariant under this symmetry if and only if the mass matrix M^{jk} is real.

For free fermions, we may always make the mass matrix diagonal and real via some unitary transform of fermions into each other, $\chi_i(x) \to U_i^{\ j}\chi_j(x)$. Consequently, the free Weyl fermions always have a CP symmetry, but its action on the original (un-transformed) spinors becomes

$$\mathbf{CP} : \chi_j(\mathbf{x}, t) \ \to \ \sum_k C_j^{\ k}\, \sigma_2\chi_k^*(-\mathbf{x}, +t) \tag{17}$$

for some unitary matrix C satisfying $C^\top MC = -M^*$.

⋆ For extra challenge, show such C matrix exists for any complex symmetric mass matrix M and check that (17) is indeed a symmetry of the free Lagrangian (15).
Note: for any symmetric complex matrix M there is a unitary matrix V such that $VMV^\top$ is real and diagonal.

However, for the interacting fermions, changing the basis and hence the CP action from (16) to (17) may spoil the CP symmetry of the interactions. For example, consider the weak interactions of the quarks. In terms of Weyl spinors, we have LH quark fields $u_i(x)$ and $d_i(x)$ and LH antiquark fields $\tilde{u}_i(x)$ and $\tilde{d}_i(x)$ where $i = 1, 2, 3$ is the family index and the color and spinor indices are suppressed. The (u_i, d_i) quark pairs are $SU(2)$ doublets while the LH antiquarks are singlets. Consequently, the gauge covariant Lagrangian for the Weyl fermions

$$\mathcal{L}_\chi \;=\; \sum_{\mathrm{all}\,\chi} \chi^\dagger\, i\bar\sigma^\mu D_\mu \chi \;+\; \text{mass terms} \tag{18}$$

contains couplings of the charged vector fields $W^\pm_\mu$ of the $SU(2)^\star$ to the LH quarks, namely

$$\mathcal{L}^{\text{weak}}_{\text{CC}} \;=\; -g_2 W^+_\mu \times u^{i\dagger}\bar\sigma^\mu d_i \;-\; g_2 W^-_\mu \times d^{i\dagger}\bar\sigma^\mu u_i \tag{19}$$

(implicit sum over i); there are other interaction terms involving the neutral weak field Z^0_μ and the EM field A_μ, but for now let's focus on the couplings of the charged weak fields $W^\pm_\mu$.

The mass matrix for all the Weyl fields is restricted by gauge symmetries and by $M^\top = M$ to have general form

$$M \;=\; \begin{array}{c c} & \begin{array}{cccc} u & \tilde u & d & \tilde d \end{array} \\ \begin{array}{c} u \\ \tilde u \\ d \\ \tilde d \end{array} & \begin{pmatrix} 0 & m_u & 0 & 0 \\ m_u^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & m_d \\ 0 & 0 & m_d^\top & 0 \end{pmatrix} \end{array} \tag{20}$$

for some general complex 3×3 matrices m_u and m_d; physically, they are Dirac mass matrices for the up-type and down-type quarks. Thus, in 3×3 matrix form

$$\mathcal{L}_{\text{mass}} \;=\; u^\top m_u \sigma_2\, \tilde u \;+\; d^\top m_d \sigma_2\, \tilde d \;+\; u^\dagger m_u^* \sigma_2\, \tilde u^* \;+\; d^\dagger m_d^* \sigma_2\, \tilde d^*. \tag{21}$$

We may always render both m_u and m_d matrices real and diagonal by changing the bases

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$\star$ In terms of the three W^a_μ fields of the $SU(2)$, two combination $W^\pm_\mu = W^1_\mu \pm iW^2_\mu$ have electric charges $\pm e$. The remaining W^3_μ field is electrically neutral; it mixes up with the $U(1)$ gauge field B_μ to make the EM field A_μ and the Z^0_μ field. I'll explain how this works later in class. For now, all we need are the $W^\pm_\mu$ fields and their couplings to the quarks.

for each type of spinor fields,

$$u_i \;\to\; \sum_j (U_u)_i^{\,j} u_j\,, \quad \tilde{u}_i \sum_j \to\; (\tilde{U}_u)_i^{\,j} \tilde{u}_j\,, \quad d_i \;\to\; \sum_j (U_d)_i^{\,j} d_j\,, \quad \tilde{d}_i \sum_j \to\; (\tilde{U}_d)_i^{\,j} \tilde{d}_j\,, \tag{22}$$

for some unitary 3×3 matrices U_u, $\tilde{U}_u$, U_d, and $\tilde{U}_d$. But in general we need $U_u \neq U_d$, which spoils the (u_i, d_i) pairing of the LH quarks into week doublets. Consequently, the couplings of the LH quarks in the mass eigenbasis to the charged weak fields $W_\mu^\pm(x)$ becomes

$$\begin{aligned} \mathcal{L}_{\mathrm{CC}}^{\mathrm{weak}} &= -g_2 W_\mu^+ \times u^\dagger V \bar{\sigma}^\mu d \;-\; g_2 W_\mu^- \times d^\dagger V^\dagger \bar{\sigma}^\mu u \\ &= -g_2 W_\mu^+ \times \sum_{ij} V_i^{\,j}\, u^{i\dagger} \bar{\sigma}_\mu d_j \;-\; g_2 W_\mu^- \times \sum_{ij} V_j^{*\,i}\, d^{j\dagger} \bar{\sigma}_\mu u_i \end{aligned} \tag{23}$$

where $V = U_u^\dagger U_d \neq 1$ is the so-called Cabibbo–Kobayashi–Maskawa matrix. This matrix is very important to the weak interaction phenomenology as it governs all kinds of flavor-changing (or rather family-changing) processes such as decays of strange mesons via $s \to u + W^- \to u + \bar{u} + d$ or $\to u + \mu^- + \bar{\nu}_\mu$. The CKM matrix also leads to CP violation; in fact, all the currently observed CP-violating processes can be explained by the CKM matrix being complex.

(f) Show that the weak interactions (23) of $W_\mu^\pm$ to the *charged weak currents* are CP-symmetric when the CKM matrix has real matrix elements but a complex CKM matrix breaks the CP symmetry.

FYI, the CP acts on the charged vector fields as

$$\mathbf{CP} : W_0^\pm(\mathbf{x}, t) \;\to\; -W_0^\mp(-\mathbf{x}, +t), \quad \mathbf{W}^\pm(\mathbf{x}, t) \;\to\; +\mathbf{W}^\mp(-\mathbf{x}, +t). \tag{24}$$