

## Bogolyubov Transform

Given some kind of annihilation and creation operators  $\hat{a}_{\mathbf{k}}$  and  $\hat{a}_{\mathbf{k}}^\dagger$  which satisfy the bosonic commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger] = 0, \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}, \mathbf{k}'}, \quad (1)$$

we may define new operators  $\hat{b}_{\mathbf{k}}$  and  $\hat{b}_{\mathbf{k}}^\dagger$  according to

$$\hat{b}_{\mathbf{k}} = \cosh(t_{\mathbf{k}})\hat{a}_{\mathbf{k}} + \sinh(t_{\mathbf{k}})\hat{a}_{-\mathbf{k}}^\dagger, \quad \hat{b}_{\mathbf{k}}^\dagger = \cosh(t_{\mathbf{k}})\hat{a}_{\mathbf{k}}^\dagger + \sinh(t_{\mathbf{k}})\hat{a}_{-\mathbf{k}} \quad (2)$$

for some arbitrary real parameters  $t_{\mathbf{k}} = t_{-\mathbf{k}}$ . These new operators  $\hat{b}_{\mathbf{k}}$  and  $\hat{b}_{\mathbf{k}}^\dagger$  satisfy the same the same bosonic commutation relations as the  $\hat{a}_{\mathbf{k}}$  and the  $\hat{a}_{\mathbf{k}}^\dagger$ :

$$[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}] = [\hat{b}_{\mathbf{k}}^\dagger, \hat{b}_{\mathbf{k}'}^\dagger] = 0, \quad [\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}, \mathbf{k}'}. \quad (3)$$

The Bogolyubov transform — replacing the ‘original’ creation and annihilation operators  $\hat{a}_{\mathbf{k}}^\dagger$  and  $\hat{a}_{\mathbf{k}}$  with the ‘transformed’ operators  $\hat{b}_{\mathbf{k}}^\dagger$  and  $\hat{b}_{\mathbf{k}}$  — is useful for diagonalizing quadratic Hamiltonians of the form

$$\hat{H} = \sum_{\mathbf{k}} A_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{k}} B_{\mathbf{k}} \left( \hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger \right) \quad (4)$$

where for all momenta  $\mathbf{k}$ ,  $A_{\mathbf{k}} = A_{-\mathbf{k}}$ ,  $B_{\mathbf{k}} = B_{-\mathbf{k}}$ , and  $A_{\mathbf{k}} > |B_{\mathbf{k}}|$ . Indeed, for a suitable choice of the  $t_{\mathbf{k}}$  parameters,

$$\hat{H} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} + \text{const} \quad \text{where } \omega_{\mathbf{k}} = \sqrt{A_{\mathbf{k}}^2 - B_{\mathbf{k}}^2}. \quad (5)$$

Moreover,  $\hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} - \hat{b}_{-\mathbf{k}}^\dagger \hat{b}_{-\mathbf{k}} = \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} - \hat{a}_{-\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}$  and consequently

$$\hat{\mathbf{P}} \equiv \sum_{\mathbf{k}} \mathbf{k} \times \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} = \sum_{\mathbf{k}} \mathbf{k} \times \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}}. \quad (6)$$

Proof of (3):

Combining definitions (2) with commutation relations (1), we immediately calculate

$$[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}] = \cosh(t_{\mathbf{k}}) \sinh(t_{\mathbf{k}'}) \delta_{\mathbf{k}, -\mathbf{k}'} - \sinh(t_{\mathbf{k}}) \cosh(t_{\mathbf{k}'}) \delta_{-\mathbf{k}, \mathbf{k}'} = 0 \quad (7)$$

where the second equality follows from  $t_{\mathbf{k}} = t_{\mathbf{k}'}$  for  $\mathbf{k} = -\mathbf{k}'$ . Likewise,  $[\hat{b}_{\mathbf{k}}^\dagger, \hat{b}_{\mathbf{k}'}^\dagger] = 0$ . Finally,

$$\begin{aligned} [\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}^\dagger] &= \cosh(t_{\mathbf{k}}) \cosh(t_{\mathbf{k}'}) \delta_{\mathbf{k}, \mathbf{k}'} - \sinh(t_{-\mathbf{k}}) \sinh(t_{-\mathbf{k}'}) \delta_{-\mathbf{k}, -\mathbf{k}'} \\ &= \delta_{\mathbf{k}, \mathbf{k}'} \left( \cosh^2(t_{\mathbf{k}}) - \sinh^2(t_{\mathbf{k}}) = 1 \right). \end{aligned} \quad (8)$$

In other words, the  $\hat{b}_{\mathbf{k}}$  and  $\hat{b}_{\mathbf{k}}^\dagger$  operators satisfy the same bosonic commutations relations

$$[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}] = 0, \quad [\hat{b}_{\mathbf{k}}^\dagger, \hat{b}_{\mathbf{k}'}^\dagger] = 0, \quad [\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}, \mathbf{k}'} . \quad (9)$$

as the original  $\hat{a}_{\mathbf{k}}$  and  $\hat{a}_{\mathbf{k}}^\dagger$  operators. *Q.E.D.*

Proof of (5):

Applying eqs. (2) twice, we immediately obtain

$$\hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} = \cosh^2(t_{\mathbf{k}}) \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \cosh(t_{\mathbf{k}}) \sinh(t_{\mathbf{k}}) (\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger + \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}}) + \sinh^2(t_{\mathbf{k}}) (\hat{a}_{-\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}} + 1). \quad (10)$$

Next, we use  $t_{-\mathbf{k}} = t_{\mathbf{k}}$  to combine

$$\begin{aligned} \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} + \hat{b}_{-\mathbf{k}}^\dagger \hat{b}_{-\mathbf{k}} &= \left( \cosh^2(t_{\mathbf{k}}) + \sinh^2(t_{\mathbf{k}}) = \cosh(2t_{\mathbf{k}}) \right) \times (\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \hat{a}_{-\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}) \\ &+ \left( 2 \cosh(t_{\mathbf{k}}) \sinh(t_{\mathbf{k}}) = \sinh(2t_{\mathbf{k}}) \right) \times (\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger + \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}}) + \text{const.} \end{aligned} \quad (11)$$

Finally, for  $\omega_{-\mathbf{k}} \equiv \omega_{\mathbf{k}}$  we have

$$\begin{aligned} \sum_{\mathbf{k}} \omega_{\mathbf{k}} \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} &= \frac{1}{2} \sum_{\mathbf{k}} \omega_{\mathbf{k}} (\hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} + \hat{b}_{-\mathbf{k}}^\dagger \hat{b}_{-\mathbf{k}}) \\ &= \sum_{\mathbf{k}} \omega_{\mathbf{k}} \cosh(2t_{\mathbf{k}}) \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{k}} \omega_{\mathbf{k}} \sinh(2t_{\mathbf{k}}) (\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger + \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}}) + \text{const.} \end{aligned} \quad (12)$$

Consequently, the Hamiltonian (4) can be “diagonalized” in terms of the transformed creation

/ annihilation operators (2), provided we can find  $\omega_{\mathbf{k}} = \omega_{-\mathbf{k}}$  and  $t_{\mathbf{k}} = t_{-\mathbf{k}}$  such that

$$\omega_{\mathbf{k}} \cosh(2t_{\mathbf{k}}) = A_{\mathbf{k}} \quad \text{and} \quad \omega_{\mathbf{k}} \sinh(2t_{\mathbf{k}}) = B_{\mathbf{k}}. \quad (13)$$

These equations are easy to solve, and the solution exists as long as  $A_{\mathbf{k}} = A_{-\mathbf{k}}$ ,  $B_{\mathbf{k}} = B_{-\mathbf{k}}$ , and  $A_{\mathbf{k}} > |B_{\mathbf{k}}|$ , namely

$$t_{\mathbf{k}} = \frac{1}{2} \operatorname{artanh} \frac{B_{\mathbf{k}}}{A_{\mathbf{k}}} \quad \text{and} \quad \omega_{\mathbf{k}} = \sqrt{A_{\mathbf{k}}^2 - B_{\mathbf{k}}^2}. \quad (14)$$

*Q.E.D.*

Proof of (6):

Using eq. (10) and  $t_{-\mathbf{k}} = t_{\mathbf{k}}$ , we immediately see that

$$\hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} - \hat{b}_{-\mathbf{k}}^\dagger \hat{b}_{-\mathbf{k}} = \left( \cosh^2(t_{\mathbf{k}}) - \sinh^2(t_{\mathbf{k}}) = 1 \right) \times (\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} - \hat{a}_{-\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}). \quad (15)$$

Consequently,

$$\begin{aligned} \hat{\mathbf{P}} &= \sum_{\mathbf{k}} \mathbf{k} \times \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} = \sum_{\mathbf{k}} (-\mathbf{k}) \times \hat{a}_{-\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}} \\ &= \frac{1}{2} \sum_{\mathbf{k}} \mathbf{k} \times (\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} - \hat{a}_{-\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}) \\ &= \frac{1}{2} \sum_{\mathbf{k}} \mathbf{k} \times (\hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} - \hat{b}_{-\mathbf{k}}^\dagger \hat{b}_{-\mathbf{k}}) \\ &= \sum_{\mathbf{k}} \mathbf{k} \times \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}}. \end{aligned} \quad (16)$$

*Q.E.D.*