Dirac Matrices and Lorentz Spinors

Background: In 3D, the spinor $j = \frac{1}{2}$ representation of the Spin(3) rotation group is constructed from the Pauli matrices σ^x, σ^y, and σ^k, which obey both commutation and anticommutation relations

$$[\sigma^i, \sigma^j] \;=\; 2i\epsilon^{ijk}\sigma^k \quad \text{and} \quad \{\sigma^i, \sigma^j\} \;=\; 2\delta^{ij} \times \mathbf{1}_{2\times 2}\,. \tag{1}$$

Consequently, the spin matrices

$$\mathbf{S} \;=\; -\tfrac{i}{2}\boldsymbol{\sigma} \times \boldsymbol{\sigma} \;=\; \tfrac{1}{2}\boldsymbol{\sigma} \tag{2}$$

commute with each other as angular momenta, $[S^i, S^j] = i\epsilon^{ijk}S^k$, so they represent the generators of the rotation group. Moreover, under finite rotations $R(\phi, \mathbf{n})$ represented by

$$M(R) \;=\; \exp\bigl(-i\phi\mathbf{n}\cdot\mathbf{S}\bigr), \tag{3}$$

the spin matrices transform into each other as components of a 3–vector,

$$M^{-1}(R) S^i M(R) \;=\; R^{ij} S^j. \tag{4}$$

In this note, I shall generalize this construction to the *Dirac spinor* representation of the Lorentz symmetry $\mathrm{Spin}(3,1)$.

Dirac Matrices are 4 anti-commuting 4×4 matrices γ^μ,

$$\gamma^\mu\gamma^\nu \;+\; \gamma^\nu\gamma^\mu \;=\; 2g^{\mu\nu} \times \mathbf{1}_{4\times 4}\,. \tag{5}$$

The specific form of these matrices is not important — as long as they obey the anticommutation relations (5) — and different books use different conventions. In my class I shall follow the same convention as the Peskin & Schroeder textbook, namely the Weyl convention where in 2×2 block notations

$$\gamma^0 \;=\; \begin{pmatrix} 0 & \mathbf{1}_{2\times 2} \\ \mathbf{1}_{2\times 2} & 0 \end{pmatrix}, \qquad \vec{\gamma} \;=\; \begin{pmatrix} 0 & +\vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}. \tag{6}$$

Note that the γ^0 matrix is hermitian while the γ^1, γ^2, and γ^3 matrices are anti-hermitian.

Lorentz spin matrices.

Given the Dirac matrices obeying the anticommutation relations (5), we may define the spin matrices as

$$S^{\mu\nu} \;=\; -S^{\nu\mu} \;\stackrel{\text{def}}{=}\; \tfrac{i}{4}[\gamma^\mu,\gamma^\nu]. \tag{7}$$

These matrices obey the same commutation relations as the generators $\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$ of the continuous Lorentz group. Moreover, their commutation relations with the Dirac matrices γ^μ are similar to the commutation relations of the $\hat{J}^{\mu\nu}$ with a Lorentz vector such as $\hat{P}^\mu$.

Lemma:

$$[\gamma^\lambda, S^{\mu\nu}] \;=\; ig^{\lambda\mu}\gamma^\nu \;-\; ig^{\lambda\nu}\gamma^\mu. \tag{8}$$

Proof: Combining the definition (7) of the spin matrices as commutators with the anticommutation relations (5), we have

$$\gamma^\mu\gamma^\nu \;=\; \tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\} \;+\; \tfrac{1}{2}[\gamma^\mu,\gamma^\nu] \;=\; g^{\mu\nu}\times\mathbf{1}_{4\times4} \;-\; 2iS^{\mu\nu}. \tag{9}$$

Since the unit matrix commutes with everything, we have

$$[X, S^{\mu\nu}] \;=\; \tfrac{i}{2}[X,\gamma^\mu\gamma^\nu] \qquad \text{for any matrix } X, \tag{10}$$

and the commutator on the RHS may often be obtained from the Leibniz rules for the commutators or anticommutators:

$$\begin{aligned} [A,BC] \;&=\; [A,B]C \;+\; B[A,C] \;=\; \{A,B\}C \;-\; B\{A,C\}, \\ \{A,BC\} \;&=\; [A,B]C \;+\; B\{A,C\} \;=\; \{A,B\}C \;-\; B[A,C]. \end{aligned} \tag{11}$$

In particular,

$$[\gamma^\lambda,\gamma^\mu\gamma^\nu] \;=\; \{\gamma^\lambda,\gamma^\mu\}\gamma^\nu \;-\; \gamma^\mu\{\gamma^\lambda,\gamma^\nu\} \;=\; 2g^{\lambda\mu}\gamma^\nu \;-\; 2g^{\lambda\nu}\gamma^\mu \tag{12}$$

and hence

$$[\gamma^\lambda, S^{\mu\nu}] \;=\; \tfrac{i}{2}[\gamma^\lambda,\gamma^\mu\gamma^\nu] \;=\; ig^{\lambda\mu}\gamma^\nu \;-\; ig^{\lambda\nu}\gamma^\mu. \tag{13}$$

Quod erat demonstrandum.

Theorem: The $S^{\mu\nu}$ matrices commute with each other like Lorentz generators,

$$\left[S^{\kappa\lambda}, S^{\mu\nu}\right] \;=\; ig^{\lambda\mu}S^{\kappa\nu} \;-\; ig^{\kappa\nu}S^{\mu\lambda} \;-\; ig^{\lambda\nu}S^{\kappa\mu} \;+\; ig^{\kappa\mu}S^{\nu\lambda}. \tag{14}$$

<u>Proof</u>: Again, we use the Leibniz rule and eq. (9):

$$\begin{aligned}
\left[\gamma^\kappa\gamma^\lambda, S^{\mu\nu}\right] \;&=\; \gamma^\kappa\left[\gamma^\lambda, S^{\mu\nu}\right] \;+\; \left[\gamma^\kappa, S^{\mu\nu}\right]\gamma^\lambda \\
&=\; \gamma^\kappa\big(ig^{\lambda\mu}\gamma^\nu \;-\; ig^{\lambda\nu}\gamma^\mu\big) \;+\; \big(ig^{\kappa\mu}\gamma^\nu \;-\; ig^{\kappa\nu}\gamma^\mu\big)\gamma^\lambda \\
&=\; ig^{\lambda\mu}\gamma^\kappa\gamma^\nu \;-\; ig^{\kappa\nu}\gamma^\mu\gamma^\lambda \;-\; ig^{\lambda\nu}\gamma^\kappa\gamma^\mu \;+\; ig^{\kappa\mu}\gamma^\nu\gamma^\lambda \\
&=\; ig^{\lambda\mu}\big(g^{\kappa\nu} - 2iS^{\kappa\nu}\big) \;-\; ig^{\kappa\nu}\big(g^{\lambda\mu} + 2iS^{\lambda\mu}\big) \\
&\qquad -\; ig^{\lambda\nu}\big(g^{\kappa\mu} - 2iS^{\kappa\mu}\big) \;+\; ig^{\kappa\mu}\big(g^{\lambda\nu} + 2iS^{\lambda\nu}\big) \\
&=\; 2g^{\lambda\mu}S^{\kappa\nu} \;-\; 2g^{\kappa\nu}S^{\lambda\mu} \;-\; 2g^{\lambda\nu}S^{\kappa\mu} \;+\; 2g^{\kappa\mu}S^{\lambda\nu},
\end{aligned} \tag{15}$$

and hence

$$\left[S^{\kappa\lambda}, S^{\mu\nu}\right] \;=\; \tfrac{i}{2}\left[\gamma^\kappa\gamma^\lambda, S^{\mu\nu}\right] \;=\; ig^{\lambda\mu}S^{\kappa\nu} \;-\; ig^{\kappa\nu}S^{\mu\lambda} \;-\; ig^{\lambda\nu}S^{\kappa\mu} \;+\; ig^{\kappa\mu}S^{\nu\lambda}. \tag{16}$$

Quod erat demonstrandum.

In light of this theorem, the $S^{\mu\nu}$ matrices *represent* the Lorentz generators $\hat{J}^{\mu\nu}$ in a 4-component spinor multiplet.

Finite Lorentz transforms:

Any continuous Lorentz transform — a rotation, or a boost, or a product of a boost and a rotation — obtains from exponentiating an infinitesimal symmetry

$$X'^\mu \;=\; X^\mu \;+\; \epsilon^{\mu\nu}X_\nu \tag{17}$$

where the infinitesimal $\epsilon^{\mu\nu}$ matrix is antisymmetric when both indices are raised (or both lowered), $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$. Thus, the $L^\mu_{\ \nu}$ matrix of any continuous Lorentz transform is a matrix exponential

$$L^\mu_{\ \nu} \;=\; \exp(\Theta)^\mu_{\ \nu} \;\equiv\; \delta^\mu_\nu \;+\; \Theta^\mu_{\ \nu} \;+\; \tfrac{1}{2}\Theta^\mu_{\ \lambda}\Theta^\lambda_{\ \nu} \;+\; \tfrac{1}{6}\Theta^\mu_{\ \lambda}\Theta^\lambda_{\ \kappa}\Theta^\kappa_{\ \nu} \;+\; \cdots \tag{18}$$

of some matrix Θ that becomes antisymmetric when both of its indices are raised or lowered, $\Theta^{\mu\nu} = -\Theta^{\nu\mu}$. Note however that in the matrix exponential (18), the first index of Θ is raised while the second index is lowered, so the antisymmetry condition becomes $(g\Theta)^\top = -(g\Theta)$ instead of $\Theta^\top = -\Theta$.

The Dirac spinor representation of the finite Lorentz transform (18) is the 4×4 matrix

$$M_D(L) \;=\; \exp\bigl(-\tfrac{i}{2}\,\Theta_{\alpha\beta}S^{\alpha\beta}\bigr). \tag{19}$$

The group law for such matrices

$$\forall L_1, L_2 \in \mathrm{SO}^+(3,1), \quad M_D(L_2L_1) \;=\; M_D(L_2)M_D(L_1) \tag{20}$$

follows automatically from the $S^{\mu\nu}$ satisfying the commutation relations (14) of the Lorentz generators, so I am not going to prove it. Instead, let me show that when the Dirac matrices γ^μ are sandwiched between the $M_D(L)$ and its inverse, they transform into each other as components of a Lorentz 4–vector,

$$M_D^{-1}(L)\gamma^\mu M_D(L) \;=\; L^\mu_{\ \nu}\gamma^\nu. \tag{21}$$

This formula makes the Dirac equation transform covariantly under the Lorentz transforms.

<u>Proof:</u> In light of the exponential form (19) of the matrix $M_D(L)$ representing a finite Lorentz transform in the Dirac spinor multiplet, let's use the multiple commutator formula (AKA the Hadamard Lemma): for any 2 matrices F and H,

$$\exp(-F)H\exp(+F) \;=\; H \;+\; \bigl[H,F\bigr] \;+\; \tfrac{1}{2}\bigl[\bigl[H,F\bigr],F\bigr] \;+\; \tfrac{1}{6}\bigl[\bigl[\bigl[H,F\bigr],F\bigr],F\bigr] \;+\; \cdots. \tag{22}$$

In particular, let $H = \gamma^\mu$ while $F = -\frac{i}{2}\,\Theta_{\alpha\beta}S^{\alpha\beta}$ so that $M_D(L) = \exp(+F)$ and $M_D^{-1}(L) = \exp(-F)$. Consequently,

$$M_D^{-1}(L)\gamma^\mu M_D(L) \;=\; \gamma^\mu \;+\; \bigl[\gamma^\mu,F\bigr] \;+\; \tfrac{1}{2}\bigl[\bigl[\gamma^\mu,F\bigr],F\bigr] \;+\; \tfrac{1}{6}\bigl[\bigl[\bigl[\gamma^\mu,F\bigr],F\bigr],F\bigr] \;+\; \cdots \tag{23}$$

where all the multiple commutators turn out to be linear combinations of the Dirac matrices.

Indeed, the single commutator here is

$$\left[\gamma^\mu,F\right] \;=\; -\tfrac{i}{2}\Theta_{\alpha\beta}\left[\gamma^\mu,S^{\alpha\beta}\right] \;=\; \tfrac12\Theta_{\alpha\beta}\bigl(g^{\mu\alpha}\gamma^\beta\,-\,g^{\mu\beta}\gamma^\alpha\bigr) \;=\; \Theta_{\alpha\beta}\,g^{\mu\alpha}\gamma^\beta \;=\; \Theta^\mu{}_\lambda\gamma^\lambda, \tag{24}$$

while the multiple commutators follow by iterating this formula:

$$\bigl[\left[\gamma^\mu,F\right],F\bigr] \;=\; \Theta^\mu{}_\lambda\bigl[\gamma^\lambda,F\bigr] \;=\; \Theta^\mu{}_\lambda\Theta^\lambda{}_\nu\gamma^\nu, \qquad \Bigl[\bigl[\left[\gamma^\mu,F\right],F\bigr],F\Bigr] \;=\; \Theta^\mu{}_\lambda\Theta^\lambda{}_\rho\Theta^\rho{}_\nu\gamma^\nu,\ldots. \tag{25}$$

Combining all these commutators as in eq. (23), we obtain

$$\begin{aligned} M_D^{-1}\gamma^\mu M_D \;&=\; \gamma^\mu \;+\; \left[\gamma^\mu,F\right] \;+\; \tfrac12\bigl[\left[\gamma^\mu,F\right],F\bigr] \;+\; \tfrac16\Bigl[\bigl[\left[\gamma^\mu,F\right],F\bigr],F\Bigr] \;+\;\cdots\\ &=\; \gamma^\mu \;+\; \Theta^\mu{}_\nu\gamma^\nu \;+\; \tfrac12\,\Theta^\mu{}_\lambda\Theta^\lambda{}_\nu\gamma^\nu \;+\; \tfrac16\,\Theta^\mu{}_\lambda\Theta^\lambda{}_\rho\Theta^\rho{}_\nu\gamma^\nu \;+\;\cdots\\ &=\; \Bigl(\delta^\mu_\nu \,+\, \Theta^\mu{}_\nu \,+\, \tfrac12\Theta^\mu{}_\lambda\Theta^\lambda{}_\nu \,+\, \tfrac16\Theta^\mu{}_\lambda\Theta^\lambda{}_\rho\Theta^\rho{}_\nu \,+\,\cdots\Bigr)\gamma^\nu\\ &\equiv\; L^\mu{}_\nu\gamma^\nu. \end{aligned} \tag{26}$$

Quod erat demonstrandum.

Dirac Equation

The Dirac spinor field $\Psi(x)$ has 4 complex components $\Psi_\alpha(x)$ arranged in a column vector

$$\Psi(x) \;=\; \begin{pmatrix} \Psi_1(x)\\ \Psi_2(x)\\ \Psi_3(x)\\ \Psi_4(x) \end{pmatrix}. \tag{27}$$

Under continuous Lorentz symmetries $x'^\mu = L^\mu{}_\nu x^\nu$, the spinor field transforms as

$$\Psi'(x') \;=\; M_D(L)\Psi(x). \tag{28}$$

The classical field equation for the free spinor field is the Dirac equation — a first-order differential equation

$$\bigl(i\gamma^\mu\partial_\mu \,-\, m\bigr)\Psi(x) \;=\; 0. \tag{29}$$

The Dirac equation implies the Klein–Gordon equation for each component $\Psi_\alpha(x)$. Indeed, if $\Psi(x)$ obey the Dirac equation, then

$$\big(-i\gamma^\nu\partial_\nu\,-\,m\big)\big(i\gamma^\mu\partial_\mu\,-\,m\big)\Psi(x)\;=\;0 \tag{30}$$

where the differential operator on the LHS is the Klein–Gordon $m^2+\partial^2$ times a unit matrix. Indeed,

$$\big(-i\gamma^\nu\partial_\nu - m\big)\big(i\gamma^\mu\partial_\mu - m\big)\;=\;m^2+\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu=\;m^2+\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\nu\partial_\mu\;=\;m^2+g^{\mu\nu}\partial_\nu\partial_\mu\,. \tag{31}$$

The Dirac equation transforms covariantly under the Lorentz symmetries — its LHS transforms exactly like the spinor field itself.

<u>Proof:</u> Note that since the Lorentz symmetries involve the x^μ coordinates as well as the spinor field components, the LHS of the Dirac equation becomes

$$\big(i\gamma^\mu\partial'_\mu\,-\,m\big)\Psi'(x') \tag{32}$$

where

$$\partial'_\mu\;\equiv\;\frac{\partial}{\partial x'^\mu}\;=\;\frac{\partial x^\nu}{\partial x'^\mu}\times\frac{\partial}{\partial x^\nu}\;=\;\big(L^{-1}\big)_\mu^{\ \nu}\times\partial_\nu\,. \tag{33}$$

Consequently,

$$\partial'_\mu\Psi'(x')\;=\;\big(L^{-1}\big)_\mu^{\ \nu}\times M_D(L)\,\partial_\nu\Psi(x) \tag{34}$$

and hence

$$\gamma^\mu\partial'_\mu\Psi'(x')\;=\;\big(L^{-1}\big)_\mu^{\ \nu}\times\gamma^\mu M_D(L)\,\partial_\nu\Psi(x). \tag{35}$$

But according to eq. (23),

$$\begin{aligned} M_D^{-1}(L)\gamma^\mu M_D(L)\;=\;L^\mu_{\ \nu}\gamma^\nu\quad&\Longrightarrow\quad\gamma^\mu M_D(L)\;=\;L^\mu_{\ \nu}\times M_D(L)\gamma^\nu\\ &\Longrightarrow\quad\big(L^{-1}\big)_\mu^{\ \nu}\times\gamma^\mu M_D(L)\;=\;M_D(L)\gamma^\nu, \end{aligned} \tag{36}$$

so

$$\gamma^\mu\partial'_\mu\Psi'(x')\;=\;M_D(L)\times\gamma^\nu\partial_\nu\Psi(x). \tag{37}$$

Altogether,

$$\big(i\gamma^\mu\partial_\mu\,-\,m\big)\Psi(x)\;\xrightarrow[\text{Lorentz}]{}\;\big(i\gamma^\mu\partial'_\mu\,-\,m\big)\Psi'(x')\;=\;M_D(L)\times\big(i\gamma^\mu\partial_\mu\,-\,m\big)\Psi(x), \tag{38}$$

which proves the covariance of the Dirac equation. *Quod erat demonstrandum.*