

Dirac Matrices and Lorentz Spinors

Background: In 3D, the spinor $j = \frac{1}{2}$ representation of the Spin(3) rotation group is constructed from the Pauli matrices σ^x , σ^y , and σ^z , which obey both commutation and anticommutation relations

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k \quad \text{and} \quad \{\sigma^i, \sigma^j\} = 2\delta^{ij} \times \mathbf{1}_{2 \times 2}. \quad (1)$$

Consequently, the spin matrices

$$\mathbf{S} = -\frac{i}{2}\boldsymbol{\sigma} \times \boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\sigma} \quad (2)$$

commute with each other as angular momenta, $[S^i, S^j] = i\epsilon^{ijk}S^k$, so they represent the generators of the rotation group. Moreover, under finite rotations $R(\phi, \mathbf{n})$ represented by

$$M(R) = \exp(-i\phi\mathbf{n} \cdot \mathbf{S}), \quad (3)$$

the spin matrices transform into each other as components of a 3-vector,

$$M^{-1}(R)S^iM(R) = R^{ij}S^j. \quad (4)$$

In this note, I shall generalize this construction to the *Dirac spinor* representation of the Lorentz symmetry Spin(3, 1).

Dirac Matrices are 4 anti-commuting 4×4 matrices γ^μ ,

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times \mathbf{1}_{4 \times 4}. \quad (5)$$

The specific form of these matrices is not important — as long as they obey the anticommutation relations (5) — and different books use different conventions. In my class I shall follow the same convention as the Peskin & Schroeder textbook, namely the Weyl convention where in 2×2 block notations

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_{2 \times 2} \\ \mathbf{1}_{2 \times 2} & 0 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & +\vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}. \quad (6)$$

Note that the γ^0 matrix is hermitian while the γ^1 , γ^2 , and γ^3 matrices are anti-hermitian.

Lorentz spin matrices.

Given the Dirac matrices obeying the anticommutation relations (5), we may define the spin matrices as

$$S^{\mu\nu} = -S^{\nu\mu} \stackrel{\text{def}}{=} \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \quad (7)$$

These matrices obey the same commutation relations as the generators $\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$ of the continuous Lorentz group. Moreover, their commutation relations with the Dirac matrices γ^μ are similar to the commutation relations of the $\hat{J}^{\mu\nu}$ with a Lorentz vector such as \hat{P}^μ .

Lemma:

$$[\gamma^\lambda, S^{\mu\nu}] = ig^{\lambda\mu}\gamma^\nu - ig^{\lambda\nu}\gamma^\mu. \quad (8)$$

Proof: Combining the definition (7) of the spin matrices as commutators with the anticommutation relations (5), we have

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = g^{\mu\nu} \times \mathbf{1}_{4 \times 4} - 2iS^{\mu\nu}. \quad (9)$$

Since the unit matrix commutes with everything, we have

$$[X, S^{\mu\nu}] = \frac{i}{2}[X, \gamma^\mu\gamma^\nu] \quad \text{for any matrix } X, \quad (10)$$

and the commutator on the RHS may often be obtained from the **Leibniz rules for the commutators or anticommutators**:

$$\begin{aligned} [A, BC] &= [A, B]C + B[A, C] = \{A, B\}C - B\{A, C\}, \\ \{A, BC\} &= [A, B]C + B\{A, C\} = \{A, B\}C - B[A, C]. \end{aligned} \quad (11)$$

In particular,

$$[\gamma^\lambda, \gamma^\mu\gamma^\nu] = \{\gamma^\lambda, \gamma^\mu\}\gamma^\nu - \gamma^\mu\{\gamma^\lambda, \gamma^\nu\} = 2g^{\lambda\mu}\gamma^\nu - 2g^{\lambda\nu}\gamma^\mu \quad (12)$$

and hence

$$[\gamma^\lambda, S^{\mu\nu}] = \frac{i}{2}[\gamma^\lambda, \gamma^\mu\gamma^\nu] = ig^{\lambda\mu}\gamma^\nu - ig^{\lambda\nu}\gamma^\mu. \quad (13)$$

Quod erat demonstrandum.

Theorem: The $S^{\mu\nu}$ matrices commute with each other like Lorentz generators,

$$[S^{\kappa\lambda}, S^{\mu\nu}] = ig^{\lambda\mu}S^{\kappa\nu} - ig^{\kappa\nu}S^{\mu\lambda} - ig^{\lambda\nu}S^{\kappa\mu} + ig^{\kappa\mu}S^{\nu\lambda}. \quad (14)$$

Proof: Again, we use the Leibniz rule and eq. (9):

$$\begin{aligned} [\gamma^\kappa\gamma^\lambda, S^{\mu\nu}] &= \gamma^\kappa [\gamma^\lambda, S^{\mu\nu}] + [\gamma^\kappa, S^{\mu\nu}] \gamma^\lambda \\ &= \gamma^\kappa (ig^{\lambda\mu}\gamma^\nu - ig^{\lambda\nu}\gamma^\mu) + (ig^{\kappa\mu}\gamma^\nu - ig^{\kappa\nu}\gamma^\mu)\gamma^\lambda \\ &= ig^{\lambda\mu}\gamma^\kappa\gamma^\nu - ig^{\kappa\nu}\gamma^\mu\gamma^\lambda - ig^{\lambda\nu}\gamma^\kappa\gamma^\mu + ig^{\kappa\mu}\gamma^\nu\gamma^\lambda \\ &= ig^{\lambda\mu}(g^{\kappa\nu} - 2iS^{\kappa\nu}) - ig^{\kappa\nu}(g^{\lambda\mu} + 2iS^{\lambda\mu}) \\ &\quad - ig^{\lambda\nu}(g^{\kappa\mu} - 2iS^{\kappa\mu}) + ig^{\kappa\mu}(g^{\lambda\nu} + 2iS^{\lambda\nu}) \\ &= 2g^{\lambda\mu}S^{\kappa\nu} - 2g^{\kappa\nu}S^{\lambda\mu} - 2g^{\lambda\nu}S^{\kappa\mu} + 2g^{\kappa\mu}S^{\lambda\nu}, \end{aligned} \quad (15)$$

and hence

$$[S^{\kappa\lambda}, S^{\mu\nu}] = \frac{i}{2} [\gamma^\kappa\gamma^\lambda, S^{\mu\nu}] = ig^{\lambda\mu}S^{\kappa\nu} - ig^{\kappa\nu}S^{\mu\lambda} - ig^{\lambda\nu}S^{\kappa\mu} + ig^{\kappa\mu}S^{\nu\lambda}. \quad (16)$$

Quod erat demonstrandum.

In light of this theorem, the $S^{\mu\nu}$ matrices *represent* the Lorentz generators $\hat{J}^{\mu\nu}$ in a 4-component spinor multiplet.

Finite Lorentz transforms:

Any continuous Lorentz transform — a rotation, or a boost, or a product of a boost and a rotation — obtains from exponentiating an infinitesimal symmetry

$$X'^{\mu} = X^{\mu} + \epsilon^{\mu\nu}X_{\nu} \quad (17)$$

where the infinitesimal $\epsilon^{\mu\nu}$ matrix is antisymmetric when both indices are raised (or both lowered), $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$. Thus, the L^{μ}_{ν} matrix of any continuous Lorentz transform is a matrix exponential

$$L^{\mu}_{\nu} = \exp(\Theta)^{\mu}_{\nu} \equiv \delta^{\mu}_{\nu} + \Theta^{\mu}_{\nu} + \frac{1}{2}\Theta^{\mu}_{\lambda}\Theta^{\lambda}_{\nu} + \frac{1}{6}\Theta^{\mu}_{\lambda}\Theta^{\lambda}_{\kappa}\Theta^{\kappa}_{\nu} + \dots \quad (18)$$

of some matrix Θ that becomes antisymmetric when both of its indices are raised or lowered, $\Theta^{\mu\nu} = -\Theta^{\nu\mu}$. Note however that in the matrix exponential (18), the first index of Θ is raised

while the second index is lowered, so the antisymmetry condition becomes $(g\Theta)^\top = -(g\Theta)$ instead of $\Theta^\top = -\Theta$.

The Dirac spinor representation of the finite Lorentz transform (18) is the 4×4 matrix

$$M_D(L) = \exp\left(-\frac{i}{2} \Theta_{\alpha\beta} S^{\alpha\beta}\right). \quad (19)$$

The group law for such matrices

$$\forall L_1, L_2 \in \text{SO}^+(3,1), \quad M_D(L_2 L_1) = M_D(L_2) M_D(L_1) \quad (20)$$

follows automatically from the $S^{\mu\nu}$ satisfying the commutation relations (14) of the Lorentz generators, so I am not going to prove it. Instead, let me show that when the Dirac matrices γ^μ are sandwiched between the $M_D(L)$ and its inverse, they transform into each other as components of a Lorentz 4-vector,

$$M_D^{-1}(L) \gamma^\mu M_D(L) = L^\mu_\nu \gamma^\nu. \quad (21)$$

This formula makes the Dirac equation transform covariantly under the Lorentz transforms.

Proof: In light of the exponential form (19) of the matrix $M_D(L)$ representing a finite Lorentz transform in the Dirac spinor multiplet, let's use the multiple commutator formula (AKA the [Hadamard Lemma](#)): for any 2 matrices F and H ,

$$\exp(-F) H \exp(+F) = H + [H, F] + \frac{1}{2} [[H, F], F] + \frac{1}{6} [[[H, F], F], F] + \dots \quad (22)$$

In particular, let $H = \gamma^\mu$ while $F = -\frac{i}{2} \Theta_{\alpha\beta} S^{\alpha\beta}$ so that $M_D(L) = \exp(+F)$ and $M_D^{-1}(L) = \exp(-F)$. Consequently,

$$M_D^{-1}(L) \gamma^\mu M_D(L) = \gamma^\mu + [\gamma^\mu, F] + \frac{1}{2} [[\gamma^\mu, F], F] + \frac{1}{6} [[[\gamma^\mu, F], F], F] + \dots \quad (23)$$

where all the multiple commutators turn out to be linear combinations of the Dirac matrices.

Indeed, the single commutator here is

$$[\gamma^\mu, F] = -\frac{i}{2}\Theta_{\alpha\beta}[\gamma^\mu, S^{\alpha\beta}] = \frac{1}{2}\Theta_{\alpha\beta}(g^{\mu\alpha}\gamma^\beta - g^{\mu\beta}\gamma^\alpha) = \Theta_{\alpha\beta}g^{\mu\alpha}\gamma^\beta = \Theta^\mu_\lambda\gamma^\lambda, \quad (24)$$

while the multiple commutators follow by iterating this formula:

$$[[\gamma^\mu, F], F] = \Theta^\mu_\lambda[\gamma^\lambda, F] = \Theta^\mu_\lambda\Theta^\lambda_\nu\gamma^\nu, \quad [[[\gamma^\mu, F], F], F] = \Theta^\mu_\lambda\Theta^\lambda_\rho\Theta^\rho_\nu\gamma^\nu, \dots \quad (25)$$

Combining all these commutators as in eq. (23), we obtain

$$\begin{aligned} M_D^{-1}\gamma^\mu M_D &= \gamma^\mu + [\gamma^\mu, F] + \frac{1}{2}[[\gamma^\mu, F], F] + \frac{1}{6}[[[\gamma^\mu, F], F], F] + \dots \\ &= \gamma^\mu + \Theta^\mu_\nu\gamma^\nu + \frac{1}{2}\Theta^\mu_\lambda\Theta^\lambda_\nu\gamma^\nu + \frac{1}{6}\Theta^\mu_\lambda\Theta^\lambda_\rho\Theta^\rho_\nu\gamma^\nu + \dots \\ &= \left(\delta^\mu_\nu + \Theta^\mu_\nu + \frac{1}{2}\Theta^\mu_\lambda\Theta^\lambda_\nu + \frac{1}{6}\Theta^\mu_\lambda\Theta^\lambda_\rho\Theta^\rho_\nu + \dots\right)\gamma^\nu \\ &\equiv L^\mu_\nu\gamma^\nu. \end{aligned} \quad (26)$$

Quod erat demonstrandum.

Dirac Equation

The Dirac spinor field $\Psi(x)$ has 4 complex components $\Psi_\alpha(x)$ arranged in a column vector

$$\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \Psi_3(x) \\ \Psi_4(x) \end{pmatrix}. \quad (27)$$

Under continuous Lorentz symmetries $x'^\mu = L^\mu_\nu x^\nu$, the spinor field transforms as

$$\Psi'(x') = M_D(L)\Psi(x). \quad (28)$$

The classical field equation for the free spinor field is the Dirac equation — a first-order differential equation

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0. \quad (29)$$

The Dirac equation implies the Klein–Gordon equation for each component $\Psi_\alpha(x)$. Indeed,

if $\Psi(x)$ obey the Dirac equation, then

$$(-i\gamma^\nu \partial_\nu - m)(i\gamma^\mu \partial_\mu - m)\Psi(x) = 0 \quad (30)$$

where the differential operator on the LHS is the Klein–Gordon $m^2 + \partial^2$ times a unit matrix. Indeed,

$$(-i\gamma^\nu \partial_\nu - m)(i\gamma^\mu \partial_\mu - m) = m^2 + \gamma^\nu \gamma^\mu \partial_\nu \partial_\mu = m^2 + \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} \partial_\nu \partial_\mu = m^2 + g^{\mu\nu} \partial_\nu \partial_\mu. \quad (31)$$

The Dirac equation transforms covariantly under the Lorentz symmetries — its LHS transforms exactly like the spinor field itself.

Proof: Note that since the Lorentz symmetries involve the x^μ coordinates as well as the spinor field components, the LHS of the Dirac equation becomes

$$(i\gamma^\mu \partial'_\mu - m)\Psi'(x') \quad (32)$$

where

$$\partial'_\mu \equiv \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \times \frac{\partial}{\partial x^\nu} = (L^{-1})^\nu_\mu \times \partial_\nu. \quad (33)$$

Consequently,

$$\partial'_\mu \Psi'(x') = (L^{-1})^\nu_\mu \times M_D(L) \partial_\nu \Psi(x) \quad (34)$$

and hence

$$\gamma^\mu \partial'_\mu \Psi'(x') = (L^{-1})^\nu_\mu \times \gamma^\mu M_D(L) \partial_\nu \Psi(x). \quad (35)$$

But according to eq. (23),

$$\begin{aligned} M_D^{-1}(L) \gamma^\mu M_D(L) = L^\mu_\nu \gamma^\nu &\implies \gamma^\mu M_D(L) = L^\mu_\nu \times M_D(L) \gamma^\nu \\ &\implies (L^{-1})^\nu_\mu \times \gamma^\mu M_D(L) = M_D(L) \gamma^\nu, \end{aligned} \quad (36)$$

so

$$\gamma^\mu \partial'_\mu \Psi'(x') = M_D(L) \times \gamma^\nu \partial_\nu \Psi(x). \quad (37)$$

Altogether,

$$(i\gamma^\mu \partial_\mu - m)\Psi(x) \xrightarrow{\text{Lorentz}} (i\gamma^\mu \partial'_\mu - m)\Psi'(x') = M_D(L) \times (i\gamma^\mu \partial_\mu - m)\Psi(x), \quad (38)$$

which proves the covariance of the Dirac equation. *Quod erat demonstrandum.*