Quantization of Non-Abelian Gauge Theories

For simplicity, let's start with the pure Yang–Mills theory with some simple gauge group G. Classically, the only fields of the theory are the gauge fields $A^a_\mu(x)$ in the adjoint multiplet of G. The Euclidean Lagrangian is

$$\mathcal{L}_E \;=\; +\frac{1}{4}\sum_a \left(F^a_{\mu\nu}\right)^2 \tag{1}$$

where

$$F^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu \;-\; \partial_\nu A^a_\mu \;-\; g f^{abc} A^b_\mu A^c_\nu \tag{2}$$

are the non-abelian tension fields, g is the gauge coupling constant, and f^{abc} are the structure constants of the Lie algebra of G. That is, the generators T^a of G obey $[T^a, T^b] = i f^{abc} T^c$.

In perturbation theory we decompose the Lagrangian into quadratic, cubic, and quartic terms,

$$\begin{aligned}
\mathcal{L} \;&=\; \mathcal{L}_2 \;+\; g\mathcal{L}_3 \;+\; g^2\mathcal{L}_4\,,\\
\mathcal{L}_2 \;&=\; \tfrac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 \;=\; \tfrac{1}{2}\left(\partial_\mu A^a_\nu\right)^2 \;-\; \tfrac{1}{2}\left(\partial_\mu A^a_\mu\right)^2,\\
\mathcal{L}_3 \;&=\; -\tfrac{1}{2}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) \times f^{abc} A^b_\mu A^c_\nu\,,\\
\mathcal{L}_4 \;&=\; \tfrac{1}{4}\left(f^{abc} A^b_\mu A^c_\nu\right)^2.
\end{aligned} \tag{3}$$

In Feynman rules, the propagators should come from the quadratic part $\mathcal{L}_2$ while the vertices should come from the cubic and quartic parts. But the quadratic part here looks like $|G|$ species of photons and it suffers from exactly the same quantization problem as the QED: the Euclidean path integral over the free $A^a_\mu(x)$ fields diverges for for generic sources $J^a_\mu(x)$ and does not give us a valid propagator.

Just as in QED, the solution to this problem is to fix a gauge. That is, for every configuration $A^a_\mu(x)$ of the gauge fields, we replace it with a gauge-equivalent configuration$\tilde{A}^a_\mu(x)$ which obeys some simple constraint at every point x, for example the Landau gauge constraint

$$\partial_\mu \tilde{A}^a_\mu(x) \;\equiv\; 0 \quad \forall x,\, \forall a. \tag{4}$$

Consequently, for the path integral we have

$$
\begin{aligned}
Z[J] &= \iiint \mathcal{D}[A^a_\mu(x)]\,\exp\bigl(-S_e[A,J]\bigr) \\
&\to \iiint \mathcal{D}[A^a_\mu(x)]\,\exp\bigl(-S_e[A,J]\bigr)\times\iiint \mathcal{D}[\Lambda^a(x)]\,\Delta[\partial_\mu\tilde{A}^a_\mu(x)]\times\mathrm{Det}[FP] \qquad (5)\\
&= \iiint \mathcal{D}[\Lambda^a(x)]\iiint \mathcal{D}[A^a_\mu(x)]\,\exp\bigl(-S_e[A,J]\bigr)\times\Delta[\partial_\mu\tilde{A}^a_\mu(x)]\times\mathrm{Det}[FP]
\end{aligned}
$$

where $\tilde{A}^a_\mu(x)$ obtains from the $A^a_\mu(x)$ via the gauge transform parametrized by the $\Lambda^a(x)$ (I'll write an explicit formula in a moment), and $\mathrm{Det}[FP]$ is the Faddeev–Popov determinant,

$$
\mathrm{Det}[FP] \;=\; \mathrm{Det}\left[\frac{\delta(\partial_\mu\tilde{A}^a_\mu)}{\delta\Lambda^b}\right]. \qquad (6)
$$

Since the net YM action is gauge invariant, on the last line of eq. (5) we may replace the $S[A]$ with the $S[\tilde{A}]$. Likewise, the measure of the path integral should also be gauge invariant, so we may replace the $\mathcal{D}[A^a_\mu]$ with the $\mathcal{D}[\tilde{A}^a_\mu(x)]$, thus

$$
\begin{aligned}
Z[J] &= \iiint \mathcal{D}[\Lambda^a(x)]\iiint \mathcal{D}[\tilde{A}^a_\mu(x)]\,\exp\bigl(-S_e[\tilde{A},J]\bigr)\times\Delta[\partial_\mu\tilde{A}^a_\mu(x)]\times\mathrm{Det}[FP] \\
&= \iiint \mathcal{D}[\Lambda^a(x)]\,\hat{Z}[J] \qquad (7)\\
\text{where}\quad \hat{Z}[J] &= \iiint \mathcal{D}[\tilde{A}^a_\mu(x)]\,\exp\bigl(-S_e[\tilde{A},J]\bigr)\times\Delta[\partial_\mu\tilde{A}^a_\mu(x)]\times\mathrm{Det}[FP]. \qquad (8)
\end{aligned}
$$

But in the last integral for the $\hat{Z}$ nothing depends on the gauge transform $\Lambda^a(x)$: the integrand depends only on the gauge field $\tilde{A}^a_\mu(x)$ constrained to obey $\partial_\mu\tilde{A}^a_\mu(x)\equiv 0$, and we integrate over all such fields. Therefore, the outer integral over the $\Lambda^a(x)$ is redundant, it does nothing but introduce an overall constant factor we do not care about, so we may just as well dispense with it. In other words, we simply re-identify the $\hat{Z}$ as the partition function of the Yang–Mills theory.

All this seem to work exactly as in QED, but the devil is in the details: the non-abelian gauge transforms are more complicated, which makes the Faddeev–Popov determinant depend on the vector fields A^a_μ. Indeed, the non-abelian gauge transforms do not merely shift the A_μ by $\partial_\mu\Lambda(x)$ but also rotate the components A^a_μ into each other. For the infinitesimal gauge transform parameters $\Lambda^a(x)$,

$$\delta A^a_\mu(x) \;=\; -\partial_\mu\Lambda^a(x) \;-\; gf^{abc}\Lambda^b(x)A^c_\mu(x) \;=\; -D_\mu\Lambda^a(x), \tag{9}$$

while the finite gauge transforms are best written in matrix notations for the symmetry group G: The transform is parametrized by the x–dependent symmetry matrix $U(x) = \exp(ig\Lambda^a(x)T^a)$, while the matrix-valued vector field $\mathcal{A}_\mu(x) = gA^a_\mu(x)T^a$ transforms as

$$\mathcal{A}_\mu(x) \;\longrightarrow\; U(x)\times\mathcal{A}_\mu(x)\times U^{-1}(x) \;+\; i\partial_\mu U(x)\times U^{-1}(x). \tag{10}$$

Fortunately, the Faddeev–Popov determinant does not depend on the finite gauge transform that gets us from some original $A^a_\mu(x)$ to the $\tilde{A}^a_\mu(x)$ that obey the Landau gauge constraint. All we need are the infinitesimal variations of that gauge transform, and we can build them in two stages:

$$\begin{array}{rlll} \text{first,} & A^a_\mu(x) \underset{\text{finite}}{\longrightarrow} \hat{A}^a_\mu(x)\,, & \text{which obeys } \partial_\mu\hat{A}^a_\mu(x)\equiv 0, \\ \text{second} & \hat{A}^a_\mu(x) \underset{\text{infi}}{\longrightarrow} \tilde{A}^a_\mu(x) \;=\; \hat{A}^a_\mu(x) \;-\; D_\mu\Lambda^a(x). \end{array} \tag{11}$$

The Faddeev–Popov determinant depends only on the second stage here, thus

$$\mathrm{Det}[FP] \;=\; \mathrm{Det}\left[\frac{\delta(\partial_\mu\tilde{A}^a_\mu)}{\delta\Lambda^b}\right] \;=\; \mathrm{Det}\big[(-\partial_\mu D_\mu)^a_{\;b}\big] \tag{12}$$

Note: the differential operator here is a product of an ordinary derivative ∂_μ and a covariant derivative D_μ (in the adjoint multiplet of the gauge group), and the covariant derivative makes the FP determinant A-dependent.

To may re-implement the Faddeev–Popov determinant (12) using a fermionic path integral. Indeed, the determinant of any matrix $\mathcal{O}_{ij}$ of differential operators obtains from the fermionic path integral

$$\iiiint\mathcal{D}[\overline{\psi}_i(x)]\iiiint\mathcal{D}[\psi_j(x)]\,\exp\left(-\int d^4x_e\,\overline{\psi}_i\mathcal{O}_{ij}\psi_j\right) \;=\; \mathrm{Det}[\mathcal{O}_{ij}]. \tag{13}$$

The number of ψ_i and the $\overline{\psi}_j$ fields here depends on the matrix size of the $\mathcal{O}_{ij}$, and their indices should be of the same type. In particular, for the Faddeev–Popov determinant (12) the fermionic fields should carry the adjoint indices of the gauge symmetry, thus

$$\mathrm{Det}[FP] \;=\; \iiint\!\mathcal{D}[\bar{c}^a(x)] \iiint\!\mathcal{D}[c^a(x)]\,\exp\left(+\int d^4x_e\,\bar{c}_a\partial_\mu D_\mu c^a \;=\; -\int d^4x_e\,\partial_\mu\bar{c}_a D_\mu c^a\right). \quad (14)$$

On the other hand, since the operator $-\partial_\mu D_\mu$ does not have any Dirac indices, the fermionic fields $c^a(x)$ and $\bar{c}^a(x)$ — called the *Faddeev–Popov ghost fields* — are spinless scalar fields despite their fermionic statistics! This violates the spin-statistics theorem, so quanta of the ghost fields are not physical particles, and their Hilbert space has negative norm.

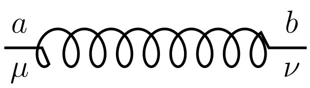
In terms of the ghosts fields, the partition function (8) for the Yang–Mills theory becomes

$$\begin{aligned} Z \;=\; &\iiint\!\mathcal{D}[A^a_\mu(x)]\,\Delta[\partial_\mu A^a_\mu(x)] \iiint\!\mathcal{D}[\bar{c}^a(x)] \iiint\!\mathcal{D}[c^a(x)] \\ &\exp\left(-\int d^4x_e\left(\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} \;-\; J^a_\mu A^a_\mu \;+\; \partial_\mu\bar{c}_a D_\mu c^a\right)\right). \end{aligned} \quad (15)$$

In other words, the quantum theory has both vector and ghost fields, its effective Euclidean Lagrangian is

$$\mathcal{L}_{\mathrm{eff}} \;=\; \tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} \;+\; \partial_\mu\bar{c}_a D_\mu c^a, \quad (16)$$

and the vector fields are constrained by the Landau gauge condition $\partial_\mu A^a_\mu(x) \equiv 0$. Thanks to this condition, the theory has well-defined vector propagators:



$$\begin{aligned} &= \frac{\delta^{ab}}{k^2}\left(\delta_{\mu\nu} \;-\; \frac{k_\mu k_\nu}{k^2}\right) && \text{(Euclidean)} \\ &= \frac{-i\delta^{ab}}{k^2+i0}\left(g_{\mu\nu} \;-\; \frac{k_\mu k_\nu}{k^2+i0}\right) && \text{(Minkowski).} \end{aligned} \quad (17)$$

Sometimes it is more convenient to use the Feynman gauge or a more general ξ gauge. To change the gauge, we proceed similar to QED. First, we modify the right hand side of the Landau gauge constraint and demand $\partial_\mu A^a_\mu(x) \equiv \omega^a(x)$ for a fixed $\omega^a(x)$. This change does not affect the Faddeev–Popov determinant, so the partition function becomes

$$Z[J,\omega] \;=\; \iiint\!\mathcal{D}[A^a_\mu(x)]\,\Delta[\partial_\mu A^a_\mu(x)-\omega^a(x)]\iiint\!\mathcal{D}[\bar c^a(x)]\iiint\!\mathcal{D}[c^a(x)]\;\exp\left(-\int\! d^4x_e\left(\tfrac14 F^a_{\mu\nu}F^a_{\mu\nu}\,-\,J^a_\mu A^a_\mu\,+\,\partial_\mu\bar c_a D_\mu c^a\right)\right). \tag{18}$$

By gauge invariance of the original theory, this partition function does not depend on $\omega^a(x)$, so we may just as well average it over the ω configurations with some Gaussian weight. In other words, we add to the theory a non-propagating auxiliary field — or rather an adjoint multiplet of auxiliary fields $\omega^a(x)$ with quadratic Lagrangian

$$\mathcal{L}_\omega \;=\; \frac{1}{2\xi}\,\omega^a\omega^a\,, \tag{19}$$

Consequently, the partition function becomes

$$\begin{aligned}
Z[J] \;&=\; \iiint\!\mathcal{D}[\omega^a(x)]\,\exp\left(\frac{-1}{2\xi}\int\! d^4x_e\,\omega^a\omega^a\right)\times Z[J,\omega]\\
&=\; \iiint\!\mathcal{D}[\omega^a(x)]\iiint\!\mathcal{D}[A^a_\mu(x)]\,\Delta[\partial_\mu A^a_\mu(x)-\omega^a(x)]\iiint\!\mathcal{D}[\bar c^a(x)]\iiint\!\mathcal{D}[c^a(x)]\\
&\qquad \exp\left(-\int\! d^4x_e\left(\tfrac14 F^a_{\mu\nu}F^a_{\mu\nu}\,-\,J^a_\mu A^a_\mu\,+\,\partial_\mu\bar c_a D_\mu c^a\,+\,\frac{1}{2\xi}\,\omega^a\omega^a\right)\right).
\end{aligned} \tag{20}$$

But in the last integral, we may use the $\Delta[A^a_\mu(x)-\omega^a(x)]$ functional to eliminate the auxiliary fields ω^a instead of constraining the vector fields, thus

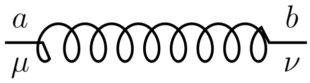
$$Z[J] \;=\; \iiint\!\mathcal{D}[A^a_\mu(x)]\iiint\!\mathcal{D}[\bar c^a(x)]\iiint\!\mathcal{D}[c^a(x)]\;\exp\left(-\int\! d^4x_e\big(\mathcal{L}_{\rm net}-J^a_\mu A^a_\mu\big)\right) \tag{21}$$

where the net Euclidean Lagrangian is now

$$\mathcal{L}_{\rm net} \;=\; \tfrac14 F^a_{\mu\nu}F^a_{\mu\nu}\;+\;\frac{1}{2\xi}\left(\partial_\mu A^a_\mu\right)^2\;+\;\partial_\mu\bar c_a D_\mu c^a \tag{22}$$

and the vector fields are no longer constrained. Instead, we have the gauge-fixing term $(\partial_\mu A^a_\mu)^2/2\xi$ in the Lagrangian. Adding this term to the quadratic part of the original YM

Lagrangian, we arrive at the ξ–gauge propagators for the vector fields,



$$
\begin{aligned}
&= \frac{\delta^{ab}}{k^2}\left(\delta_{\mu\nu} \,+\, (\xi-1)\times\frac{k_\mu k_\nu}{k^2}\right) && \text{(Euclidean)} \\
&= \frac{-i\delta^{ab}}{k^2+i0}\left(g_{\mu\nu} \,+\, (\xi-1)\times\frac{k_\mu k_\nu}{k^2+i0}\right) && \text{(Minkowski).}
\end{aligned}
\tag{23}
$$

The Feynman gauge is the special case of this gauge for $\xi = 1$.