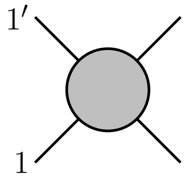
Mandelstam Variables

Consider any kind of a 2 particles → 2 particles process



(1)

The 4-momenta p_1^μ, p_2^μ, $p_1'^\mu$, and $p_2'^\mu$ of the 2 incoming and 2 outgoing particles satisfy 8 constraints: the on-shell conditions for each particle

$$p_1^2 \;=\; m_1^2, \quad p_2^2 = m_2^2, \quad p_1'^2 \;=\; m_1'^2, \quad p_2'^2 \;=\; m_2'^2, \tag{2}$$

and the net 4-momentum conservation

$$p_1^\mu \;+\; p_2^\mu \;=\; p_1'^\mu \;-\; p_2'^\mu. \tag{3}$$

Altogether, this gives us $4 \times 4 - 8 = 8$ independent momentum variables, and the number of independent Lorentz-invariant combinations of these variables is only $8 - 6 = 2$.

However, for practical purposes it's is often convenient to use 3 Lorentz-invariant variables with a fixed sum,

$$\begin{aligned} s &= (p_1 + p_2)^2 \;=\; (p_1' + p_2')^2, \\ t &= (p_1 - p_1')^2 \;=\; (p_2' - p_2)^2, \\ u &= (p_1 - p_2')^2 \;=\; (p_1' - p_2)^2. \\ s + t + u &= m_1^2 + m_2^2 + m_1'^2 + m_2'^2\,. \end{aligned} \tag{4}$$

Indeed,

$$\begin{aligned} s + t + u &= (p_1 + p_2)^2 \;+\; (p_1 - p_1')^2 \;+\; (p_1 - p_2')^2 \\ &= 3p_1^2 \;+\; p_2^2 \;+\; p_1'^2 \;+\; p_2'^2 \;+\; 2(p_1 p_2) \;-\; 2(p_1 p_1') \;-\; 2(p_1 p_2') \\ &= p_1^2 \;+\; p_2^2 \;+\; p_1'^2 \;+\; p_2'^2 + \; 2p_1 \times (p_1 + p_2 - p_1' - p_2' = 0) \\ &= p_1^2 \;+\; p_2^2 \;+\; p_1'^2 \;+\; p_2'^2 \\ &= m_1^2 \;+\; m_2^2 \;+\; m_1'^2 \;+\; m_2'^2\,. \end{aligned} \tag{5}$$

The s, t, and u are called Mandelstam variables after Stanley Mandelstam who introduced them back in 1958.

All Lorentz-invariant combinations of the four momenta p_1^μ, p_2^μ, $p_1'^\mu$, and $p_2'^\mu$ can be expressed in terms of the Mandelstam variables. For example, the Lorentz products $k^\mu k'_\mu$ of any two momenta are

$$\begin{aligned}
2(p_1p_2) &= (p_1+p_2)^2 - p_1^2 - p_2^2 = s - m_1^2 - m_2^2\,, \\
2(p_1'p_2') &= (p_1'+p_2')^2 - p_1'^2 - p_2'^2 = s - m_1'^2 - m_2'^2\,, \\
2(p_1p_1') &= p_1^2 + p_1'^2 - (p_1-p_1')^2 = m_1^2 + m_1'^2 - t, \\
2(p_2p_2') &= p_2^2 + p_2'^2 - (p_2-p_2')^2 = m_2^2 + m_2'^2 - t, \\
2(p_1p_2') &= p_1^2 + p_2'^2 - (p_1-p_2')^2 = m_1^2 + m_2'^2 - u, \\
2(p_2p_1') &= p_2^2 + p_1'^2 - (p_2-p_1')^2 = m_2^2 + m_1'^2 - u.
\end{aligned} \tag{6}$$

In particular, for an elastic scattering of 2 same-mass particles

$$\begin{aligned}
s + t + u &= 4m^2, \\
2(p_1p_2) = 2(p_1'p_2') &= s - 2m^2, \\
2(p_1p_1') = 2(p_2p_2') &= 2m^2 - t, \\
2(p_1p_2') = 2(p_2p_1') &= 2m^2 - u.
\end{aligned} \tag{7}$$

For future reference, let me give you similar formulae for the $e^-e^+ \to \mu^-\mu^+$ pair-production,

$$\begin{aligned}
s + t + u &= 2M_\mu^2 + 2m_e^2 \approx 2M_\mu^2, \\
2(p_1p_2) &= s - 2m_e^2 \approx s, \\
2(p_1'p_2') &= s - 2M_\mu^2, \\
2(p_1p_1') = 2(p_2p_2') &= M_\mu^2 + m_e^2 - t \approx M_\mu^2 - t, \\
2(p_1p_2') = 2(p_2p_1') &= M_\mu^2 + m_e^2 - u \approx M_\mu^2 - u.
\end{aligned} \tag{8}$$

and for the $e^-e^+ \to \gamma\gamma$ annihilation process $p_- + p_+ \to k_1 + k_2$,

$$\begin{aligned}
s + t + u &= 2m_e^2\,, \\
2(p_-p_+) &= s - 2m_e^2\,, \\
2(k_1k_2) &= s, \\
2(p_-k_1) = 2(p_+k_2) &= m_e^2 - t, \\
2(p_-k_2) = 2(p_+k_1) &= m_e^2 - u.
\end{aligned} \tag{9}$$