



where

$$\mathcal{D} = (p_1^2 - m^2 + i0) \times (p_2^2 - m^2 + i0) \quad (8)$$

and

$$\begin{aligned} \mathcal{N}^{\mu\nu} &= \text{tr}(\gamma^\nu(\not{p}_2 + m)\gamma^\mu(\not{p}_1 + m)) \\ &= \text{tr}(\gamma^\nu \not{p}_2 \gamma^\mu \not{p}_1) + m^2 \text{tr}(\gamma^\nu \gamma^\mu) \\ &= 4p_1^\mu p_2^\nu + 4p_1^\nu p_2^\mu - 4(p_1 p_2)g^{\mu\nu} + 4m^2 g^{\mu\nu}. \end{aligned} \quad (9)$$

In the denominator  $\mathcal{D}$ , we may combine the two factors using the Feynman's parameter trick, thus

$$\begin{aligned} \frac{1}{\mathcal{D}} &= \int_0^1 dx \frac{1}{[(1-x)p_1^2 + xp_2^2 - m^2 + ip]^2} \\ \text{where } [\dots] &= (1-x) \times p_1^2 + x \times (p_2 \equiv p_1 + k)^2 - m^2 + i0 \\ &= p_1^2 + 2x \times (p_1 k) + x \times k^2 - m^2 + i0 \\ &= (p_1 + xk)^2 + (x - x^2) \times k^2 - m^2 + i0 \\ &\equiv p^2 - \Delta(x) + i0 \end{aligned} \quad (10)$$

in terms of  $p \stackrel{\text{def}}{=} p_1 + k$  and

$$\Delta(x) \stackrel{\text{def}}{=} m^2 - x(1-x) \times k^2. \quad (11)$$

Altogether,

$$\text{the trace} = \int_0^1 dx \frac{e^2 \mathcal{N}^{\mu\nu}}{[p^2 - \Delta(x) + i0]^2} \quad (12)$$

and therefore

$$\Sigma_{1\text{loop}}^{\mu\nu}(k) = ie^2 \int_0^1 dx \int \frac{d^4 p}{(2\pi)^4} \frac{\mathcal{N}^{\mu\nu}}{[p^2 - \Delta + i0]^2}, \quad (13)$$

where in the last formula we have changed the order of integration over the loop momentum  $p_1$  and over the Feynman parameter  $x$ , and then shifted the independent momentum variable from  $p_1$  to  $p = p_1 + xk$ .

To make full use of this shift, we need to re-express the numerator  $\mathcal{N}^{\mu\nu}$  in terms of  $p$ ,  $k$ , and  $x$ . Using

$$p_1 = p - x \times k, \quad p_2 = p_1 + k = p + (1 - x) \times k, \quad (14)$$

we obtain

$$\begin{aligned} p_1^\mu p_2^\nu + p_2^\mu p_1^\nu &= 2p^\mu p^\nu + (1 - 2x) \times (p^\mu k^\nu + k^\mu p^\nu) - 2x(1 - x) \times k^\mu k^\nu, \\ (p_1 p_2) &= p^2 + (1 - 2x) \times (pk) - x(1 - x) \times k^2, \end{aligned} \quad (15)$$

and hence

$$\begin{aligned} \mathcal{N}^{\mu\nu} &= 8p^\mu p^\nu + 4(1 - 2x) \times (p^\mu k^\nu + k^\mu p^\nu) - 8x(1 - x) \times k^\mu k^\nu \\ &\quad + 4g^{\mu\nu} \times \left( m^2 - p^2 - (1 - 2x) \times (pk) + x(1 - x) \times k^2 \right). \end{aligned} \quad (16)$$

There are many terms in this expression, and it is convenient to re-organize them into 3 groups: *the good*, *the bad*, and *the odd*, thus

$$\mathcal{N}^{\mu\nu} = \mathcal{N}_{\text{good}}^{\mu\nu} + \mathcal{N}_{\text{bad}}^{\mu\nu} + \mathcal{N}_{\text{odd}}^{\mu\nu}, \quad (17)$$

where

$$\mathcal{N}_{\text{good}}^{\mu\nu} = 8x(1 - x) \times \left( g^{\mu\nu} \times k^2 - k^\mu k^\nu \right), \quad (18)$$

$$\mathcal{N}_{\text{bad}}^{\mu\nu} = 8p^\mu p^\nu + 4g^{\mu\nu} \times \left( m^2 - p^2 - x(1 - x) \times k^2 = \Delta(x) - p^2 \right), \quad (19)$$

$$\mathcal{N}_{\text{odd}}^{\mu\nu} = 4(1 - 2x) \times \left( p^\mu k^\nu + k^\mu p^\nu - g^{\mu\nu} \times (pk) \right). \quad (20)$$

The purpose of this re-organization is to extract the *good* terms (18) — which clearly have the desirable form (3) — while the remaining *bad* and *odd* terms should not contribute to the momentum integral.

Indeed, consider the *odd* terms (20) which comprise all the odd powers of the independent momentum variable  $p^\alpha$ . Consequently, under the symmetry  $p^\alpha \rightarrow -p^\alpha$  (for all 4 components

of the  $p^\alpha$ ) the  $\mathcal{N}_{\text{odd}}^{\mu\nu}$  changes its sign. On the other hand, the  $\int d^4p$  (over the whole momentum space) is invariant under this symmetry, and so is the denominator  $[p^2 - \Delta + i0]^2$ . Consequently,

$$\int \frac{d^4p}{(2\pi)^4} \frac{\mathcal{N}_{\text{odd}}^{\mu\nu}}{[p^2 - \Delta + i0]^2} \xrightarrow{p \rightarrow -p} -\text{itself} \implies \int \frac{d^4p}{(2\pi)^4} \frac{\mathcal{N}_{\text{odd}}^{\mu\nu}}{[p^2 - \Delta + i0]^2} = 0. \quad (21)$$

The momentum integral over the *bad* terms (19) also vanishes, but proving that takes more effort. First, let's Wick rotate the momentum integral from the Minkowski to the Euclidean space,

$$p^0 \rightarrow ip^4, \quad d^4p \rightarrow id^4p_E, \quad p^2 \rightarrow -p_E^2, \quad (22)$$

and hence

$$\begin{aligned} (\text{bad}) &\equiv i \int \frac{d^4p}{(2\pi)^4} \frac{\mathcal{N}_{\text{odd}}^{\mu\nu} = 8p^\mu p^\nu + 4g^{\mu\nu} \times (\Delta - p^2)}{[p^2 - \Delta + i0]^2} \\ &= - \int \frac{d^4p_E}{(2\pi)^4} \frac{8p_E^\mu p_E^\nu + 4g^{\mu\nu} \times (\Delta + p_E^2)}{[\Delta + p_E^2]^2}. \end{aligned} \quad (23)$$

Note: even in the Euclidean momentum space,  $\mu$  and  $\nu$  remain Minkowski-signature vector indices, so  $g^{\mu\nu}$  remains the Minkowski metric tensor, and  $p_E^\mu$  and  $p_E^\nu$  should be understood as  $(ip^4, p^1, p^2, p^3)$ .

Second, let's use the  $SO(4)$  symmetry of the Euclidean momentum space. Thanks to this symmetry,

$$\int \frac{d^3\Omega_p}{2\pi^2} p_E^i p_E^j = \delta^{ij} \times \frac{p_E^2}{4} \quad (24)$$

and hence for any spherically symmetric function  $f(p_E^2)$

$$\int d^4p_E f(p_E^2) \times p_E^i p_E^j = \delta^{ij} \times \int d^4p_E f(p_E^2) \times \frac{p_E^2}{4}. \quad (25)$$

Or in terms of Minkowski indices  $\mu$  and  $\nu$ ,

$$\int d^4p_E f(p_E^2) \times p_E^\mu p_E^\nu = -g^{\mu\nu} \times \int d^4p_E f(p_E^2) \times \frac{p_E^2}{4}. \quad (26)$$

More generally, in  $D$  spacetime dimensions,

$$\int d^D p_E f(p_E^2) \times p_E^\mu p_E^\nu = -g^{\mu\nu} \times \int d^D p_E f(p_E^2) \times \frac{p_E^2}{D}. \quad (27)$$

Applying this formula to the Euclidean momentum integral in eq (23) gives us

$$\begin{aligned}
(\text{bad}) &= - \int \frac{d^4 p_E}{(2\pi)^4} \frac{-g^{\mu\nu} \times (8p_E^2/D) + 4g^{\mu\nu} \times (\Delta + p_E^2)}{[\Delta + p_E^2]^2} \\
&= -g^{\mu\nu} \times \int \frac{d^4 p_E}{(2\pi)^4} \frac{4\Delta + (4 - \frac{8}{D}) \times p_E^2}{[\Delta + p_E^2]^2}.
\end{aligned} \tag{28}$$

For the moment  $D = 4$ , but we keep the dimension  $D$  as explicit parameter in order to allow for the dimensional regularization of the momentum integral. Indeed, this integral badly needs DR — or some other UV regulator — because it's quadratically divergent in  $D = 4$ . Thus, we let

$$(\text{bad}) = -g^{\mu\nu} \times \mu^{4-D} \int \frac{d^D p_E}{(2\pi)^D} \frac{4\Delta + (4 - \frac{8}{D}) \times p_E^2}{[\Delta + p_E^2]^2}, \tag{29}$$

for generic  $D$ , evaluate the integral for  $D < 2$  (which regulates the UV divergence), and then analytically continue the result back to  $D = 4 - 2\epsilon$ . As usual, to evaluate the integral for non-integer  $D$  we relate the integrand to an exponential  $\exp(-tp_E^2)$ . Specifically, we let

$$\begin{aligned}
\frac{4\Delta + (4 - \frac{8}{D}) \times p_E^2}{[\Delta + p_E^2]^2} &= \frac{(4 - \frac{8}{D})}{\Delta + p_E^2} + \frac{\frac{8}{D} \Delta}{[\Delta + p_E^2]^2} \\
&= \int_0^\infty dt \left( \left(4 - \frac{8}{D}\right) + \frac{8}{D} \Delta \times t \right) \times \exp(-t(\Delta + p_E^2)),
\end{aligned} \tag{30}$$

which leads to

$$\begin{aligned}
(\text{bad}) &= -g^{\mu\nu} \times \mu^{4-D} \int \frac{d^D p_E}{(2\pi)^D} \int_0^\infty dt \left( \left(4 - \frac{8}{D}\right) + \frac{8}{D} \Delta \times t \right) \times \exp(-t(\Delta + p_E^2)), \\
&= -g^{\mu\nu} \times \int_0^\infty dt \left( \left(4 - \frac{8}{D}\right) + \frac{8}{D} \Delta \times t \right) e^{-t\Delta} \times \mu^{4-D} \int \frac{d^D p_E}{(2\pi)^D} e^{-tp_E^2} \\
&= -g^{\mu\nu} \times \int_0^\infty dt \left( \left(4 - \frac{8}{D}\right) + \frac{8}{D} \Delta \times t \right) e^{-t\Delta} \times \mu^{4-D} (4\pi t)^{-D/2}.
\end{aligned} \tag{31}$$

The remaining integral over  $t$  here converges for  $D < 2$ , and — moracle of miracles — it happens to vanish identically for any  $D < 2$ . Indeed, up to the overall constant factor

$$-g^{\mu\nu} \mu^{4-D} (4\pi)^{-D/2},$$

$$\begin{aligned}
(\text{bad}) &\propto \int_0^\infty dt e^{-\Delta \times t} \left( \left(4 - \frac{8}{D}\right) \times t^{-D/2} + \frac{8}{D} \Delta \times t^{1-(D/2)} \right) \\
&= \left(4 - \frac{8}{D}\right) \times \Delta^{(D/2)-1} \Gamma\left(1 - \frac{D}{2}\right) + \frac{8}{D} \Delta \times \Delta^{(D/2)-2} \Gamma\left(2 - \frac{D}{2}\right) \\
&= \frac{8}{D} \times \Delta^{(D/2)-1} \times \left[ \left(\frac{D}{2} - 1\right) \times \Gamma\left(1 - \frac{D}{2}\right) + \Gamma\left(2 - \frac{D}{2}\right) \right] \\
&= \frac{8}{D} \times \Delta^{(D/2)-1} \times \left[ -y\Gamma(y) + \Gamma(y+1) \quad \text{for } y = 1 - \frac{D}{2} \right] \\
&= 0
\end{aligned} \tag{32}$$

because  $\Gamma(y+1) - y\Gamma(y) = 0$  for any  $y$ . Consequently, analytically continuing from  $D < 2$  to  $D = 4 - 2\epsilon$ , we find that the dimensionally-regulated integral (28) vanishes and hence

$$\int_{\text{DR}} \frac{d^4 p}{(2\pi)^4} \frac{\mathcal{N}_{\text{bad}}^{\mu\nu}}{[p^2 - \Delta + i0]^2} = 0. \tag{33}$$

In other words, the *bad* terms in the numerator  $\mathcal{N}^{\mu\nu}$  do not contribute to the photon's  $\Sigma_{1\text{loop}}^{\mu\nu}(k)$ .

At this point, the only terms in the the numerator  $\mathcal{N}^{\mu\nu}$  that do contribute to the integral (13) are the *good* terms (18), thus

$$\begin{aligned}
\Sigma_{1\text{loop}}^{\mu\nu}(k) &= ie^2 \int_0^1 dx \int \frac{d^4 p}{(2\pi)^4} \frac{\mathcal{N}_{\text{good}}^{\mu\nu}}{[p^2 - \Delta + i0]^2} = 8x(1-x) \times \frac{(g^{\mu\nu} k^2 - k^\mu k^\nu)}{[p^2 - \Delta + i0]^2} \\
&= 8e^2 (g^{\mu\nu} k^2 - k^\mu k^\nu) \times \int_0^1 dx x(1-x) \times \int \frac{d^4 p}{(2\pi)^4} \frac{i}{[p^2 - \Delta + i0]^2}.
\end{aligned} \tag{34}$$

In other words, the one-loop  $\Sigma^{\mu\nu}(k)$  does have the requisite form

$$\Sigma_{1\text{loop}}^{\mu\nu}(k) = (g^{\mu\nu} k^2 - k^\mu k^\nu) \times \Pi_{1\text{loop}}(k^2) \tag{35}$$

where

$$\Pi_{1\text{loop}}(k^2) = 8e^2 \int_0^1 dx x(1-x) \times \int \frac{d^4 p}{(2\pi)^4} \frac{i}{[p^2 - \Delta + i0]^2}. \tag{36}$$

It remains to evaluate the momentum integral in the last formula and then integrate over  $x$ . The momentum integral has the form we have seen before in this class, so we evaluate it

in the usual way: Wick rotate  $p$  to the Euclidean momentum space, and then dimensionally regularize the logarithmic divergence. Thus,

$$\begin{aligned}
\int \frac{d^4 p}{(2\pi)^4} \frac{i}{[p^2 - \Delta + i0]^2} &= \int \frac{d^4 p_E}{(2\pi)^4} \frac{-1}{[p_E^2 + \Delta]^2} \\
&\rightarrow -\mu^{4-D} \int \frac{d^D p_E}{(2\pi)^D} \frac{1}{[p_E^2 + \Delta]^2} \\
&= -\mu^{4-D} \int \frac{d^D p_E}{(2\pi)^D} \int_0^\infty dt t e^{-t(\Delta + p_E^2)} \\
&= -\int_0^\infty dt t e^{-\Delta \times t} \times \mu^{4-D} \int \frac{d^D p_E}{(2\pi)^D} e^{-t \times p_E^2} \\
&= -\int_0^\infty dt t e^{-\Delta \times t} \times \mu^{4-D} (4\pi t)^{-D/2} \\
&= -\frac{(4\pi\mu^2)^\epsilon}{16\pi^2} \times \int_0^\infty dt t^{1-(D/2)=\epsilon-1} e^{-\Delta \times t} \\
&= -\frac{(4\pi\mu^2)^\epsilon}{16\pi^2} \times \Gamma(\epsilon) \Delta^{-\epsilon} \\
&= -\frac{1}{16\pi^2} \times \left( \frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{\Delta} + O(\epsilon) \right)
\end{aligned} \tag{37}$$

and consequently

$$\Pi_{1\text{ loop}}(k^2) = -\frac{8e^2}{16\pi^2} \int_0^1 dx x(1-x) \times \left( \frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{\Delta(x)} \right). \tag{38}$$

Finally, using

$$\int_0^1 dx x(1-x) = \frac{1}{6} \tag{39}$$

we reduce eq. (38) to

$$\Pi_{1\text{ loop}}(k^2) = -\frac{e^2}{12\pi^2} \times \left( \frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{m^2} + I(k^2/m^2) \right) \tag{40}$$

where

$$I(k^2/m^2) = \int_0^1 dx 6x(1-x) \times \log \frac{m^2}{\Delta = m^2 - x(1-x) \times k^2} \quad (41)$$

Thus far, we have focused on the one-loop diagram (4) but ignored the counterterms. Adding the  $\delta_3$  counterterm to the picture gives us

$$\begin{aligned} i\Sigma_{\text{order } e^2}^{\mu\nu}(k) &= \text{diagram of a fermion loop with two external wavy lines} + \text{diagram of a counterterm with a wavy line and a red/blue dot} \\ &= i\Sigma_{1\text{loop}}^{\mu\nu}(k) - i\delta_3^{\text{order } e^2} \times (g^{\mu\nu} k^2 - k^\mu k^\nu), \end{aligned} \quad (42)$$

or in terms of  $\Pi(k^2)$ ,

$$\Pi^{\text{order } e^2}(k^2) = \Pi^{1\text{loop}}(k^2) - \delta_3^{\text{order } e^2}. \quad (43)$$

Consequently, by setting

$$\delta_3^{\text{order } e^2} = -\frac{e^2}{12\pi^2} \times \left( \frac{1}{\epsilon} + \text{a finite constant} \right) \quad (44)$$

we may cancel the ultraviolet divergence of the electron loop.

The finite part of the  $\delta_3$  counterterm follows from the requirement

$$\Pi^{\text{net}}(k^2 = 0) = 0. \quad (45)$$

Since  $I(0) = 0$ , this means we need

$$\delta_3^{\text{order } e^2} = -\frac{e^2}{12\pi^2} \times \left( \frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{m_e^2} \right) \quad (46)$$

which leads to

$$\Pi^{\text{order } e^2}(k^2) = -\frac{e^2}{12\pi^2} \times I(k^2/m_e^2). \quad (47)$$