# Correlation Functions in Perturbation Theory

Many aspects of quantum field theory are related to its $n$-point correlation functions

$$\mathcal{F}_n(x_1,\ldots,x_n) \stackrel{\text{def}}{=} \langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x_1)\cdots\hat{\Phi}_H(x_n)\,|\Omega\rangle \tag{1}$$

— or for theories with multiple fields $\hat{\Phi}^a$,

$$\mathcal{F}_n^{a_1,\ldots,a_n}(x_1,\ldots,x_n) \stackrel{\text{def}}{=} \langle\Omega|\,\mathbf{T}\hat{\Phi}_H^{a_1}(x_1)\cdots\hat{\Phi}_H^{a_n}(x_n)\,|\Omega\rangle\,. \tag{2}$$

Note that all the fields $\hat{\Phi}_H(x)$ here are in the Heisenberg picture so their time dependence involves the complete Hamiltonian $\hat{H}$ of the interacting theory. Likewise, $|\Omega\rangle$ is the ground state of $\hat{H}$, *i.e.* the true physical vacuum of the theory.

In perturbation theory, the correlation functions $\mathcal{F}_n$ of the interacting theory are related to the free theory's correlation functions

$$\langle 0|\,\mathbf{T}\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\cdots\text{more }\hat{\Phi}_I(z_1)\hat{\Phi}_I(z_2)\cdots|0\rangle\,. \tag{3}$$

involving additional fields $\hat{\Phi}_I(z_1)\hat{\Phi}_I(z_2)\cdots$. Note that in eq. (3) the fields are in the interaction rather than Heisenberg picture, so they evolve with time as free fields according to the free Hamiltonian $\hat{H}_0$. Likewise, $|0\rangle$ is the free theory's vacuum, *i.e.* the ground state of the free Hamiltonian $\hat{H}_0$ rather than the full Hamiltonian $\hat{H}$.

To work out the relation between (1) and (3), we start by formally relating quantum fields in the Heisenberg and the interaction pictures,

$$\hat{\Phi}_H(\mathbf{x},t) \;=\; e^{+i\hat{H}t}\hat{\Phi}_S(\mathbf{x})e^{-i\hat{H}t} \;=\; e^{+i\hat{H}t}e^{-i\hat{H}_0t}\hat{\Phi}_I(\mathbf{x},t)e^{+i\hat{H}_0t}e^{-i\hat{H}t}. \tag{4}$$

We may re-state this relation in terms of evolution operators using a formal expression for the later,

$$\hat{U}_I(t,t_0) \;=\; e^{+i\hat{H}_0t}e^{-i\hat{H}(t-t_0)}e^{-i\hat{H}_0t_0}. \tag{5}$$

Note that this formula applies for both forward and backward evolution, *i.e.* regardless of whether $t > t_0$ or $t < t_0$. In particular,

$$\hat{U}_I(t,0) \;=\; e^{+i\hat{H}_0 t}e^{-i\hat{H}t} \qquad \text{and} \qquad \hat{U}_I(0,t) \;=\; e^{+i\hat{H}t}e^{-i\hat{H}_0 t}, \tag{6}$$

which allows us to re-state eq. (4) as

$$\hat{\Phi}_H(x) \;=\; \hat{U}_I(0,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,0). \tag{7}$$

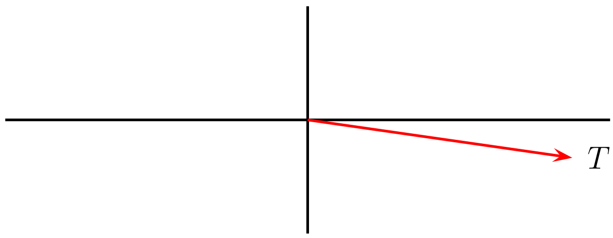
Consequently,

$$\hat{\Phi}_H(x)\hat{\Phi}_H(y) \;=\; \hat{U}_I(0,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,0) \tag{8}$$

because $\hat{U}_I(x^0,0)\hat{U}_I(0,y^0) = \hat{U}_I(x^0,y^0)$, and likewise for $n$ fields

$$\begin{aligned}\hat{\Phi}_H(x_1)\hat{\Phi}_H(x_2)\cdots\hat{\Phi}_H(x_n) \;&=\\ &= \hat{U}_I(0,x_1^0)\hat{\Phi}_I(x_1)\hat{U}_I(x_1^0,x_2^0)\hat{\Phi}_I(x_2)\cdots\hat{U}_I(x_{n-1}^0,x_n^0)\hat{\Phi}_I(x_n)\hat{U}_I(x_n^0,0).\end{aligned} \tag{9}$$

Now we need to relate the free vacuum $|0\rangle$ and the true physical vacuum $|\Omega\rangle$. Consider the state $\hat{U}_I(0,-T)\,|0\rangle$ for a complex $T$, and take the limit of $T \to (+1 - i\epsilon) \times \infty$. That is, $\operatorname{Re} T \to +\infty$, $\operatorname{Im} T \to -\infty$, but the imaginary part grows slower than the real part. Pictorially, in the complex $T$ plane,

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we go infinitely far to the right at infinitesimally small angle below the real axis.

Without loss of generality we assume the free theory has zero vacuum energy, thus $\hat{H}_0 |0\rangle = 0$ and hence

$$\hat{U}_I(0,-T)\,|0\rangle \;=\; e^{-i\hat{H}T}e^{+i\hat{H}_0 T}\,|0\rangle \;=\; e^{-i\hat{H}T}\,|0\rangle\,. \tag{11}$$

From the interacting theory's point of view, $|0\rangle$ is a superposition of eigenstates $|Q\rangle$ of the full Hamiltonian $\hat{H}$,

$$|0\rangle \;=\; \sum_Q |Q\rangle \times \langle Q|0\rangle \quad\Longrightarrow\quad e^{-i\hat{H}T}\,|0\rangle \;=\; \sum_Q |Q\rangle \times e^{-iTE_Q}\,\langle Q|0\rangle \tag{12}$$

In the $T \to (+1 - i\epsilon) \times \infty$ limit, the second sum here is dominated by the term with the lowest $E_Q$, so we look for the lowest energy eigenstate $|Q_0\rangle$ with the same quantum numbers as $|0\rangle$ (otherwise, we would have zero overlap $\langle Q_0|0\rangle$). Obviously, such $|Q_0\rangle$ is the physical vacuum $|\Omega\rangle$, so

$$\hat{U}_I(0,-T)\,|0\rangle \;\xrightarrow[T\to(+1-i\epsilon)\infty]{}\; |\Omega\rangle \times e^{-iTE_\Omega}\,\langle\Omega|0\rangle \tag{13}$$

and therefore

$$|\Omega\rangle \;=\; \lim_{T\to(+1-i\epsilon)\infty} \hat{U}_I(0,-T)\,|0\rangle \times \frac{e^{+iTE_\Omega}}{\langle\Omega|0\rangle}\,. \tag{14}$$

Likewise,

$$\langle\Omega| \;=\; \lim_{T\to(+1-i\epsilon)\infty} \frac{e^{+iTE_\Omega}}{\langle 0|\Omega\rangle} \times \langle 0|\,\hat{U}_I(+T,0)\,. \tag{15}$$

Combining eqs. (8), (14), and (15), we may now express the two-point function as

$$\langle\Omega|\,\hat{\Phi}_H(x)\hat{\Phi}_H(y)\,|\Omega\rangle \;=\; \lim_{T\to(+1-i\epsilon)\infty} C(T) \times \langle 0|\,\mathrm{Big\_Product}\,|0\rangle \tag{16}$$

where

$$C(T) \;=\; \frac{e^{2iTE_\Omega}}{|\langle 0|\Omega\rangle|^2} \tag{17}$$

is a just a coefficient, and

$$\begin{aligned}\mathrm{Big\_Product} \;&=\; \hat{U}_I(+T,0)\hat{U}_I(0,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,0)\hat{U}_I(0,-T)\\ &=\; \hat{U}_I(+T,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,-T).\end{aligned} \tag{18}$$

For $x^0 > y^0$, the last line here is in proper time order, so if we re-order the operators, the time-orderer $\mathbf{T}$ would put them back where they belong. Thus, using $\mathbf{T}$ to keep track of the operator order, we have

$$\begin{aligned}
\text{Big_Product} &= \mathbf{T}\Big(\hat{U}_I(+T,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,-T)\Big) \\
&= \mathbf{T}\Big(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\hat{U}_I(+T,x^0)\hat{U}_I(x^0,y^0)\hat{U}_I(y^0,-T)\Big) \\
&= \mathbf{T}\Big(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\hat{U}_I(+T,-T)\Big) \\
&= \mathbf{T}\left(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\exp\left(\frac{-i\lambda}{24}\int_{-T}^{+T}dt\int d^3\mathbf{z}\,\hat{\Phi}_I^4(t,\mathbf{z})\right)\right)
\end{aligned} \tag{19}$$

where the last line follows from the Dyson series for the evolution operator

$$U_I(t_f,t_i) = \mathbf{T}\text{-exp}\left(-i\int_{t_i}^{t_f}dt\,\hat{V}_I(t)\right) = \mathbf{T}\text{-exp}\left(\frac{-i\lambda}{24}\int_{t_i}^{t_f}dt\int d^3\mathbf{z}\,\hat{\Phi}_I^4(t,\mathbf{z})\right).$$

Altogether, the two-point correlation function becomes

$$\begin{aligned}
\mathcal{F}_2(x,y) &\stackrel{\text{def}}{=} \langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x)\hat{\Phi}_H(y)\,|\Omega\rangle \\
&= \lim_{T\to(+1-i\epsilon)\infty} C(T)\times\langle 0|\,\mathbf{T}\left(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\exp\left(\frac{-i\lambda}{24}\int d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle\,,
\end{aligned} \tag{20}$$

where the spacetime integral has ranges

$$\int d^4z \equiv \int_{-T}^{+T}dz^0\int_{\substack{\text{whole}\\\text{space}}}d^3\mathbf{z}\,. \tag{21}$$

Similarly, the $n$-point correlation functions can be written as

$$\begin{aligned}
\mathcal{F}_n(x_1,\ldots,x_n) &\stackrel{\text{def}}{=} \langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x_1)\cdots\hat{\Phi}_H(x_n)\,|\Omega\rangle \\
&= \lim_{T\to(+1-i\epsilon)\infty} C(T)\times\langle 0|\,\mathbf{T}\left(\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\exp\left(\frac{-i\lambda}{24}\int d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle\,.
\end{aligned} \tag{22}$$

Note that the coefficient $C(T)$ — *cf.* eq. (17) — is the same for all correlation functions.

In particular, for $n = 0$ the $\mathcal{F}_0 = \langle\Omega|\Omega\rangle = 1$, but it's also given by eq. (22), hence

$$\lim_{T\to(+1-i\epsilon)\infty} C(T) \times \langle 0|\,\mathbf{T}\left(\exp\left(\frac{-i\lambda}{24}\int d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle \;=\; 1. \tag{23}$$

This allows us to eliminate the $C(T)$ factors from eqs. (22) by taking *ratios* of the free-theory correlation functions,

$$\mathcal{F}_n(x_1,\ldots,x_n) \;=\; \lim_T \frac{\langle 0|\,\mathbf{T}\left(\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\exp\left(\frac{-i\lambda}{24}\int d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle}{\langle 0|\,\mathbf{T}\left(\exp\left(\frac{-i\lambda}{24}\int d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle}\,. \tag{24}$$

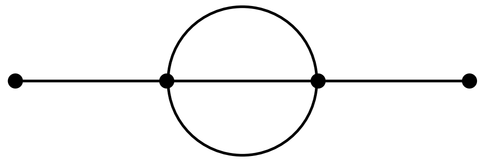
The limit here is $T \to (+1 - i\epsilon)\times\infty$, and the $T$ dependence under the limit is implicit in the ranges of the spacetime integrals, *cf.* eq. (21).

In perturbation theory, the vacuum sandwiches in the numerator and the denominator of eq. (24) can be expanded into sums of Feynman diagrams. Indeed, expanding the numerator in a power series in $\lambda$, we obtain

$$\begin{aligned}&\langle 0|\mathbf{T}\left(\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\exp\left(\frac{-i\lambda}{24}\int d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle \;=\\ &=\sum_{N=0}^{\infty}\frac{(-i\lambda)^N}{(4!)^N\,N!}\int d^4z_1\cdots\int d^4z_N\;\langle 0|\,\mathbf{T}\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\hat{\Phi}_I^4(z_1)\cdots\hat{\Phi}_I^4(z_N)\,|0\rangle\end{aligned} \tag{25}$$

where each sub-sandwich $\langle 0|\,\mathbf{T}\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\hat{\Phi}_I^4(z_1)\cdots\hat{\Phi}_I^4(z_N)\,|0\rangle$ expands into a big sum of products of $\frac{4N+n}{2}$ Feynman propagators $G_F(x_i - x_j)$, $G_F(x_i - z_j)$, or $G_F(z_i - z_j)$. We have gone through expansion back in November — here are my notes — so let me simply summarize the result in terms of the *Feynman rules for the correlation functions*:

- ⋆ A generic Feynman diagram for the $n$-point correlation function has $n$ *external vertices* $x_1,\ldots,x_n$ or valence $= 1$ plus some number $N = 0,1,2,3,\ldots$ of *internal vertices* $z_1,\ldots,Z_N$ of valence $= 4$. On the other hand, it has no external lines but only the internal lines between the vertices. Here is an example diagram with 2 external vertices, 2 internal vertices, and 5 internal lines:



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- To evaluate a diagram in coordinate space, first multiply the usual factors:

  ∗ The free propagator $G_F(z_i - z_j)$ for a line connecting vertices internal $z_i$ and $z_j$, and likewise for lines connecting an internal vertex $z_i$ to an external vertex $x_j$, or two external vertices $x_i$ and $x_j$.

  ∗ $(-i\lambda)$ factor for each internal vertex.

  ∗ The combinatorial factor 1/#symmetries of the diagram (including the trivial symmetry).

  Second, integrate $\int d^4z$ over each internal vertex location; the integration range is as in eq. (21). But do not integrate over the external vertices — their location's $x_1, \ldots, x_n$ are the arguments of the $n$-point correlation function $\mathcal{F}_n(x_1, \ldots, x_n)$.

- To calculate the numerator of eq. (24) to order $\lambda^{N_{\max}}$, sum over *all* diagrams with $n$ external vertices, $N \leq N_{\max}$ internal vertices, and any pattern of lines respecting the valences of all the vertices.

At this point, we are summing over all kinds of diagrams, connected or disconnected, and even the vacuum bubbles are allowed. However, similar to what we had back in November, the vacuum bubbles can be factored out:

$$\sum(\text{all diagrams}) \;=\; \sum\begin{pmatrix}\text{diagrams without}\\ \text{vacuum bubbles}\end{pmatrix} \times \sum\begin{pmatrix}\text{vacuum bubbles}\\ \text{without external vertices}\end{pmatrix}. \tag{27}$$

Moreover, the vacuum bubble factor here is the same for all the free-theory vacuum sandwiches

$$\langle 0|\,\mathbf{T}\left(\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\exp\left(\frac{-i\lambda}{24}\int d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle$$

in the numerators of eqs. (24) for all the correlation functions, and also in the $n = 0$ sandwich

$$\langle 0 | \mathbf{T} \left( \exp \left( \frac{-i\lambda}{24} \int d^4 z \, \hat{\Phi}_I^4(z) \right) \right) | 0 \rangle = \sum \left( \begin{array}{c} \text{vacuum bubbles} \\ \text{without external vertices} \end{array} \right) \tag{28}$$

in the all the denominators. This means that the vacuum bubbles simply cancel out from the correlation functions! In other words,

$$\mathcal{F}_n(x_1, \ldots, x_n) = \sum \left( \begin{array}{c} \text{Feynman diagrams with} \\ n \text{ external vertices } x_1, \ldots x_n \\ \text{and without vacuum bubbles} \end{array} \right). \tag{29}$$

Besides reducing the number of diagrams we need to calculate, the cancellation of the vacuum bubbles leads to another simplification: Instead of evaluating each diagram for a finite $T$, taking the ratio of two sums diagrams, and only then taking the $T \to (+1 - i\epsilon)\infty$ limit, we may now take that limit directly for each diagram . In practice, this means integrating each $\int d^4 z_i$ over the whole Minkowski spacetime instead of a limited time range from $-T$ to $+$ as in eq. (21). Consequently, when we Fourier transform the Feynman rules from the coordinate space to the momentum space, we end up with the usual momentum-conservation factors $(2\pi)^4 \delta^{(4)}(\pm q_1^{\pm} q_2 \pm q_3 \pm q_4)$ at each internal vertex instead of something much more complicated.

So here are the *momentum-space Feynman rules for the correlation functions:*

- Since all the lines are internal, assign a variable momentum $q_i^\mu$ to each line and specify the direction of this momentum flow (from which vertex to which vertex).

∗ Each line carries a propagator $\dfrac{i}{q^2 - m^2 + i0}$ .

∗ Each external vertex $x$ carries a factor $e^{+iqx}$ or $e^{-iqx}$, depending on whether the momentum $q$ flows into or out from the vertex.

∗ Each internal vertex carries factor $(-i\lambda) \times (2\pi)^4 \delta^{(4)}(\pm q_1^{\pm} q_2 \pm q_3 \pm q_4)$.

∗ Overall combinatorial factor 1/#symmetries for the whole diagram.

- Multiply all these factors together, then integrate over all the momenta $q_i^\mu$.

For example, the diagram (26) evaluates to

$$\begin{aligned}
\mathcal{F}_2(x,y)\ \supset\ &\frac{1}{6}\int\frac{d^4q_1}{(2\pi)^4}\cdots\int\frac{d^4q_5}{(2\pi)^4}\prod_{i=1}^{5}\frac{i}{q_i^2-m^2+i\epsilon}\times e^{-iq_1x}\times e^{+iq_2y}\times\\
&\qquad\times(-i\lambda)(2\pi)^4\delta^{(4)}(q_1-q_3-q_4-q_5)\times\\
&\qquad\times(-i\lambda)(2\pi)^4\delta^{(4)}(q_3+q_4+q_5-q_2)\\
=\ &\frac{-i\lambda^2}{6}\int\frac{d^4q_1}{(2\pi)^4}\,e^{-iq_1(x-y)}\times\left(\frac{1}{q_1^2-m^2+i\epsilon}\right)^2\times\\
&\qquad\times\iint\frac{d^4q_3\,d^4q_4}{(2\pi)^8}\,\frac{1}{q_3^2-m^2+i\epsilon}\times\frac{1}{q_4^2-m^2+i\epsilon}\times\\
&\qquad\times\frac{1}{(q_5=q_1-q_3-q_4)^2-m^2+i\epsilon}
\end{aligned}\tag{30}$$

Note: as defined in eq. (1), the correlation functions $\mathcal{F}_n(x_1,\ldots,x_n)$ obtain by summing *all* Feynman diagrams without vacuum bubbles, *cf.* eq. (29). Both the connected and the disconnected diagrams are included, as long as each connected part of a disconnected diagram has some external vertices. However, the disconnected diagrams' contributions can be re-summed in terms of correlation functions of fewer fields. Indeed, let's define the *connected correlation functions*

$$\mathcal{F}_n^{\rm conn}(x_1,\ldots,x_n)\ =\ \sum\begin{pmatrix}\textit{connected}\text{ Feynman diagrams}\\ \text{with } n \text{ external vertices}\end{pmatrix}.\tag{31}$$

Then the original $\mathcal{F}_n$ functions can be obtained from these via *cluster expansion:*

$$\begin{aligned}
\mathcal{F}_2(x,y)\ &=\ \mathcal{F}_2^{\rm conn}(x,y),\\
\mathcal{F}_4(x,y,x,w)\ &=\ \mathcal{F}_4^{\rm conn}(x,y,z,w)\ +\ \mathcal{F}_2^{\rm conn}(x,y)\times\mathcal{F}_2^{\rm conn}(z,w)\\
&\quad+\ \mathcal{F}_2^{\rm conn}(x,z)\times\mathcal{F}_2^{\rm conn}(y,w)\ +\ \mathcal{F}_2^{\rm conn}(x,w)\times\mathcal{F}_2^{\rm conn}(y,z),\\
\mathcal{F}_6(x,y,x,u,v,w)\ &=\ \mathcal{F}_6^{\rm conn}(x,y,z,u,v,w)\\
&\quad+\ \Big(\mathcal{F}_2^{\rm conn}(x,y)\times\mathcal{F}_4^{\rm conn}(z,u,v,w)\ +\ \text{permutations}\Big)\\
&\quad+\ \Big(\mathcal{F}_2^{\rm conn}(x,y)\times\mathcal{F}_2^{\rm conn}(z,u)\times\mathcal{F}_2^{\rm conn}(v,w)\ +\ \text{permutations}\Big),\\
&\textit{etc., etc.}
\end{aligned}\tag{32}$$