0. First of all, refresh your memory of special relativity. Make sure you understand index summation conventions in Minkowski or Euclidean spaces. If you don't understand (or have hard time deciphering) expressions such as $B_i = \epsilon_{ijk}\partial_j A_k$ (in 3 space dimensions) or $\partial_\mu F^{\mu\nu} = J^\nu$ (in the Minkowski spacetime), *get up to speed ASAP* or you would not be able to follow the class.

1. Consider a *massive* relativistic vector field $A^\mu(x)$ with the Lagrangian density

$$\mathcal{L} \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \;+\; \tfrac{1}{2}m^2\,A_\mu A^\mu \;-\; A^\mu J_\mu \tag{1}$$

where $c = \hbar = 1$, $F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$, and the current $J^\mu(x)$ is a fixed source for the $A^\mu(x)$ field. Note that because of the mass term, the Lagrangian (1) is *not* gauge invariant.

   (a) Derive the Euler–Lagrange field equations for the massive vector field $A^\mu(x)$.

   (b) Show that this field equation *does not require* current conservation; however, if the current happens to satisfy $\partial_\mu J^\mu = 0$, then the field $A^\mu(x)$ satisfies

$$\partial_\mu A^\mu \;=\; 0 \qquad \text{and} \qquad (\partial^2 + m^2)A^\mu = J^\mu. \tag{2}$$

2. In spacetimes of higher dimensions $D > 4$ there are antisymmetric-tensor fields analogous to the EM-like vector fields; such tensor fields play important roles in supergravity and string theory.

   For example, consider a free 2-index antisymmetric tensor field $B_{\mu\nu}(x) \equiv -B_{\nu\mu}(x)$, where μ and ν are D-dimensional Lorentz indices running from 0 to $D-1$. To be precise, $B_{\mu\nu}(x)$ is the *tensor potential*, analogous to the electromagnetic vector potential $A_\mu(x)$. The analog of the EM tension fields $F_{\mu\nu}(x)$ is is the 3–index tension tensor

$$H_{\lambda\mu\nu}(x) \;=\; \partial_\lambda B_{\mu\nu} \;+\; \partial_\mu B_{\nu\lambda} \;+\; \partial_\nu B_{\lambda\mu}\,. \tag{3}$$

   (a) Show that this tensor is totally antisymmetric in all 3 indices.

(b) Show that regardless of the Lagrangian, the H fields satisfy Jacobi identities

$$\tfrac{1}{6}\partial_{[\kappa}H_{\lambda\mu\nu]} \equiv \partial_\kappa H_{\lambda\mu\nu} - \partial_\lambda H_{\mu\nu\kappa} + \partial_\mu H_{\nu\kappa\lambda} - \partial_\nu H_{\kappa\lambda\mu} = 0. \tag{4}$$

(c) The Lagrangian for the free $B_{\mu\nu}(x)$ fields is simply

$$\mathcal{L}(B, \partial B) = \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} \tag{5}$$

where $H_{\lambda\mu\nu}$ are as in eq. (3). Treating the $B_{\mu\nu}(x)$ as $\frac{1}{2}D(D-1)$ independent fields, derive their equations of motion.

Similar to the EM fields, the $B_{\mu\nu}$ fields are subject to *gauge transforms*

$$B'_{\mu\nu}(x) = B_{\mu\nu}(x) + \partial_\mu\Lambda_\nu(x) - \partial_\nu\Lambda_\mu(x) \tag{6}$$

where $\Lambda_\mu(x)$ is an arbitrary vector field.

(d) Show that the tension fields $H_{\lambda\mu\nu}(x)$ — and hence the Lagrangian (5) — are invariant under such gauge transforms.

In spacetimes of sufficiently high dimensions D, one may have similar tensor fields with more indices. Generally, the potentials form a p-index totally antisymmetric tensor $C_{\mu_1\mu_2\cdots\mu_p}(x)$, the tensions form a $p+1$ index tensor

$$\begin{aligned} G_{\mu_1\mu_2\cdots\mu_{p+1}} &= \frac{1}{p!}\,\partial_{[\mu_1}C_{\mu_2\cdots\mu_p\mu_{p+1}]} \\ &\equiv \partial_{\mu_1}C_{\mu_2\cdots\mu_{p+1}} - \partial_{\mu_2}C_{\mu_1\mu_3\cdots\mu_{p+1}} + \cdots + (-1)^p\partial_{\mu_{p+1}}C_{\mu_1\cdots\mu_p}\,, \end{aligned} \tag{7}$$

also totally antisymmetric in all its indices, and the Lagrangian is

$$\mathcal{L}(C, \partial C) = \frac{(-1)^p}{2(p+1)!}\,G_{\mu_1\mu_2\cdots\mu_{p+1}}G^{\mu_1\mu_2\cdots\mu_{p+1}}. \tag{8}$$

(e) Derive the Jacobi identities and the equations of motion for the G fields.

(f) Show that the tension fields $G_{\mu_1\mu_2\cdots\mu_{p+1}}(x)$ — and hence the Lagrangian (8) — are invariant under gauge transforms of the potentials $C_{\mu_1\mu_2\cdots\mu_p}(x)$ which act as

$$C'_{\mu_1\mu_2\cdots\mu_p}(x) \;=\; C_{\mu_1\mu_2\cdots\mu_p}(x) \;+\; \frac{1}{(p-1)!}\,\partial_{[\mu_1}\Lambda_{\mu_2\cdots\mu_p]}(x) \tag{9}$$

where $\Lambda_{\mu_2\cdots\mu_p}(x)$ is an arbitrary $(p-1)$-index tensor field (totally antisymmetric).

3. According to the Noether theorem, a translationally invariant system of classical fields $\phi_a(x)$ has a conserved stress-energy tensor

$$T^{\mu\nu}_{\text{Noether}} = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi^a \;-\; g^{\mu\nu}\,\mathcal{L}. \tag{10}$$

Actually, to assure the symmetry of the stress-energy tensor, $T^{\mu\nu} = T^{\nu\mu}$ (which is necessary for the angular momentum conservation), one sometimes has to add a total divergence,

$$T^{\mu\nu} \;=\; T^{\mu\nu}_{\text{Noether}} \;+\; \partial_\lambda\mathcal{K}^{\lambda\mu\,\nu}, \tag{11}$$

where $\mathcal{K}^{\lambda\mu\,\nu} \;\equiv\; -\mathcal{K}^{\mu\lambda\,\nu}$ is some 3–index Lorentz tensor antisymmetric in its first two indices.

(a) Show that regardless of the specific form of the $\mathcal{K}^{\lambda\mu\,\nu}(\phi,\partial\phi)$ as a function of the fields and their derivatives, we have

$$\begin{aligned} \partial_\mu T^{\mu\nu} \;&=\; \partial_\mu T^{\mu\nu}_{\text{Noether}} \;=\text{(hopefully)}=\; 0 \\ \text{and}\quad P^\mu_{\text{net}} \;&\equiv\; \int\! d^3\mathbf{x}\, T^{0\mu} \;=\; \int\! d^3\mathbf{x}\, T^{0\mu}_{\text{Noether}}\,. \end{aligned} \tag{12}$$

Note: Assume that all the fields go to zero for $|\mathbf{x}| \to \infty$ fast enough that all the surface integrals over the boundary of 3D space vanish when we push the boundary to infinity.

For the scalar fields, real or complex, $T^{\mu\nu}_{\text{Noether}}$ is properly symmetric and one simply has $T^{\mu\nu} = T^{\mu\nu}_{\text{Noether}}$. Unfortunately, the situation is more complicated for the vector, tensor or spinor fields. To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) \;=\; -\tfrac{1}{4}\, F_{\mu\nu} F^{\mu\nu} \tag{13}$$

where A_μ is a real vector field and $F_{\mu\nu} \stackrel{\rm def}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$.

(b) Write down $T^{\mu\nu}_{\rm Noether}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.

(c) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T^{\mu\nu}_{\rm EM} \;=\; -F^{\mu\lambda} F^{\nu}_{\ \lambda} \;+\; \tfrac{1}{4}\, g^{\mu\nu}\, F_{\kappa\lambda} F^{\kappa\lambda}. \tag{14}$$

Show that this expression indeed has form (11) for some $\mathcal{K}^{\lambda\mu\nu}$.

(d) Write down the components of the stress-energy tensor (14) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density and pressure.

Finally, consider the electromagnetic fields coupled to the electric current J^μ of some charged "matter" fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate $P^\mu_{\rm EM}$ and $P^\mu_{\rm mat}$. Consequently, we should have

$$\partial_\mu T^{\mu\nu}_{\rm net} \;=\; 0 \qquad \text{for} \quad T^{\mu\nu}_{\rm net} \;=\; T^{\mu\nu}_{\rm EM} \;+\; T^{\mu\nu}_{\rm mat} \tag{15}$$

but generally $\partial_\mu T^{\mu\nu}_{\rm EM} \neq 0$ and $\partial_\mu T^{\mu\nu}_{\rm mat} \neq 0$.

(e) Use Maxwell's equations to show that

$$\partial_\mu T^{\mu\nu}_{\rm EM} \;=\; -F^{\nu\lambda} J_\lambda \tag{16}$$

and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_λ according to

$$\partial_\mu T^{\mu\nu}_{\rm mat} \;=\; +F^{\nu\lambda} J_\lambda. \tag{17}$$