1. First, do problem **3** from the previous homework set.

2. Continuing problem 1, consider the EM fields coupled to a specific model of charged matter, namely a complex scalar field $\Phi(x) \neq \Phi^*(x)$ of electric charge $q \neq 0$. Altogether, the net Lagrangian for the A^μ, Φ, and Φ^* fields is

$$\mathcal{L}_{\text{net}} = D^\mu\Phi^* D_\mu\Phi - m^2\Phi^*\Phi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{1}$$

where

$$D_\mu\Phi = (\partial_\mu + iqA_\mu)\Phi \quad \text{and} \quad D_\mu\Phi^* = (\partial_\mu - iqA_\mu)\Phi^* \tag{2}$$

are the *covariant* derivatives.

(a) Write down the equation of motion for all fields in a covariant from. Also, write down the electric current

$$J^\mu \stackrel{\text{def}}{=} -\frac{\partial\mathcal{L}}{\partial A_\mu} \tag{3}$$

in a manifestly gauge-invariant form and verify its conservation, $\partial_\mu J^\mu = 0$ (as long as the scalar fields satisfy their equations of motion).

(b) Write down the Noether stress-energy tensor for the whole system and show that

$$T^{\mu\nu}_{\text{net}} \equiv T^{\mu\nu}_{\text{EM}} + T^{\mu\nu}_{\text{mat}} = T^{\mu\nu}_{\text{Noether}} + \partial_\lambda\mathcal{K}^{\lambda\mu\nu}, \tag{4}$$

where $T^{\mu\nu}_{\text{EM}}$ is exactly as for the free EM fields (see eq. (14) from the previous homework set), the improvement tensor $\mathcal{K}^{\lambda\mu\nu} = -\mathcal{K}^{\mu\lambda\nu}$ is also exactly as for the free EM (see problem **1**), and

$$T^{\mu\nu}_{\text{mat}} = D^\mu\Phi^* D^\nu\Phi + D^\nu\Phi^* D^\mu\Phi - g^{\mu\nu}\big(D_\lambda\Phi^* D^\lambda\Phi - m^2\Phi^*\Phi\big). \tag{5}$$

Note: although the improvement tensor $\mathcal{K}^{\lambda\mu\nu}$ for the EM + matter system is the same as for the free EM fields, in presence of an electric current J^μ its derivative

$\partial_\lambda \mathcal{K}^{\lambda\mu\nu}$ contains an extra $J^\mu A^\nu$ term. Pay attention to this term — it is important for obtaining a gauge-invariant stress-energy tensor (5) for the scalar field.

(c) Use the scalar fields' equations of motion and the non-commutativity of covariant derivatives

$$[D_\mu, D_\nu]\Phi \;=\; iqF_{\mu\nu}\Phi, \qquad [D_\mu, D_\nu]\Phi^* \;=\; -iqF_{\mu\nu}\Phi^* \tag{6}$$

to show that

$$\partial_\mu T^{\mu\nu}_{\text{mat}} \;=\; +F^{\nu\lambda}J_\lambda \tag{7}$$

and therefore the *net* stress-energy tensor (4) is conserved, *cf.* problem **1**(e).

3. Finally, consider the Noether currents of an internal rather than translational symmetry. Let $\Phi_a(x)$ be N complex scalar fields — of similar masses and electric charges — which interact with each other and with the EM fields A^μ according to the Lagrangian

$$\mathcal{L} \;=\; \sum_a D_\mu\Phi_a^*\, D^\mu\Phi_a \;-\; m^2\sum_a \Phi_a^*\Phi_a \;-\; \frac{\lambda}{4}\left(\sum_a \Phi_a^*\Phi_a\right)^2 \;-\; \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{8}$$

where the $D_\mu\Phi_a$ and $D_\mu\Phi_a^*$ are as in eq. (2) — these derivatives are covariant with respect to the local $U(1)$ phase symmetry rather than the global $U(N)$ symmetry which is the subject of this problem. In any case, what you need for this problem is not the covariance of the D_μ derivatives but the facts that (A) they involve the EM fields $A_\mu(x)$ and (B) all the Φ_a fields have the same electric charge $+q$ while all the conjugate fields Φ_a^* have the opposite charge $-q$.

(a) Check that this Lagrangian has a $U(N)$ global symmetry:

$$\Phi_a(x) \;\to\; \sum_b U^{ab}\Phi_b(x), \qquad \Phi_a^*(x) \;\to\; \sum_b \Phi_b^*(x)U^\dagger_{ba}\,, \qquad A^\mu(x)\ \text{invariant} \tag{9}$$

for any *unitary* $N\times N$ matrix $U = \|U_{ab}\|$.

(b) Infinitesimal $U(N)$ symmetries have form $U = 1 + i\epsilon T$ — *i.e.*, $U_{ab} = \delta_{ab} + i\epsilon T_{ab}$ — for *hermitian* matrices T. Derive the Noether currents for such symmetries, the show that they can be packaged into an hermitian matrix of currents

$$J^\mu_{ab} \;=\; i\Phi_a D^\mu \Phi^*_b \;-\; i\Phi^*_b D^\mu \Phi_a \;=\; \left(J^\mu_{ba}\right)^*. \tag{10}$$

(c) Verify the conservation of the currents (10).

The scalar potential in the Lagrangian (8) has a bigger symmetry than the $U(N)$, namely the $SO(2N)$ which rotates the real and imaginary parts of the $\Phi_a(x)$ fields as if they were $2N$ unrelated real fields. But the $SO(2N)$ symmetry outside of the $U(N)$ do not commute with the local $U(1)$ symmetry of the charged fields and hence do not preserve their couplings to the EM fields.

(d) Work this out.

The infinitesimal form of an $SO(2N)$ symmetry outside of the $U(N)$ is

$$\delta\Phi_a(x) \;=\; \epsilon C_{ab}\Phi^*_b(x), \qquad \delta\Phi^*_a(x) \;=\; \epsilon C^*_{ab}\Phi_b(x) \tag{11}$$

for a complex antisymmetric matrix $C_{ab} = -C_{ba}$.

(e) Write down the Noether current for such a would-be-symmetry and show that it is NOT conserved (unless $A^\mu = 0$).