1. In class, I have focused on the *fundamental multiplet* of the local $SU(N)$ symmetry, *i.e.*, a set of N fields (complex scalars or Dirac fermions) which transform as a complex N–vector,

$$\Psi'(x) \;=\; U(x)\Psi(x) \quad i.e. \quad \Psi'_i(x) \;= \sum_j U_i^{\;j}(x)\Psi_j(x), \quad i,j=1,2,\ldots,N \tag{1}$$

where $U(x)$ is an x–dependent unitary $N\times N$ matrix, $\det U(x)\equiv 1$. Now consider N^2-1 real fields $\Phi^a(x)$ forming an *adjoint multiplet*: In matrix form

$$\Phi(x) \;=\; \sum_a \Phi^a(x)\times\frac{\lambda^a}{2} \tag{2}$$

is a traceless hermitian $N\times N$ matrix which transforms under the local $SU(N)$ symmetry as

$$\Phi'(x) \;=\; U(x)\Phi(x)U^\dagger(x). \tag{3}$$

Note that this transformation law preserves the $\Phi^\dagger=\Phi$ and $\mathrm{tr}(\Phi)=0$ conditions.

The covariant derivatives D_μ act on an adjoint multiplet of fields multiplet as

$$D_\mu\Phi(x) \;=\; \partial_\mu\Phi(x) \;+\; i[\mathcal{A}_\mu(x),\Phi(x)] \;\equiv\; \partial_\mu\Phi(x) \;+\; i\mathcal{A}_\mu(x)\Phi(x) \;-\; i\Phi(x)\mathcal{A}_\mu(x), \tag{4}$$

or in components

$$D_\mu\Phi^a(x) = \;\partial_\mu\Phi_a(x) \;-\; f^{abc}\mathcal{A}^b_\mu(x)\Phi^c(x). \tag{5}$$

(a) Verify that these derivatives are indeed covariant — the $D_\mu\Phi(x)$ transforms under the local $SU(N)$ symmetry exactly like the $\Phi(x)$ itself.

(b) Verify the Leibniz rule for covariant derivatives of matrix products. Let $\Phi(x)$ and $\Xi(x)$ be two adjoint multiplets while $\Psi(x)$ is a fundamental multiplet and $\Psi^\dagger(x)$ is its hermitian conjugate (row vector of Ψ_i^*). Show that

$$\begin{aligned} D_\mu(\Phi\Xi) &= (D_\mu\Phi)\Xi + \Phi(D_\mu\Xi), \\ D_\mu(\Phi\Psi) &= (D_\mu\Phi)\Psi + \Phi(D_\mu\Psi), \\ D_\mu(\Psi^\dagger\Xi) &= (D_\mu\Psi^\dagger)\Xi + \Psi^\dagger(D_\mu\Xi). \end{aligned} \tag{6}$$

(c) Show that for an adjoint multiplet $\Phi(x)$,

$$[D_\mu, D_\nu]\Phi(x) = i[\mathcal{F}_{\mu\nu}(x), \Phi(x)] = ig[F_{\mu\nu}(x), \Phi(x)] \tag{7}$$

or in components $[D_\mu, D_\nu]\Phi^a(x) = -gf^{abc}F^b_{\mu\nu}(x)\Phi^c(x)$.

- In my notations A_μ and $F_{\mu\nu}$ are canonically normalized fields while $\mathcal{A}_\mu = gA_\mu$ and $\mathcal{F}_{\mu\nu} = gF_{\mu\nu}$ are normalized by the symmetry action.

In class, I have argued (using covariant derivatives) that the tension fields $\mathcal{F}_{\mu\nu}(x)$ themselves transform according to eq. (3). In other words, the $\mathcal{F}^a_{\mu\nu}(x)$ form an adjoint multiplet of the $SU(N)$ symmetry group.

(d) Verify the $\mathcal{F}'_{\mu\nu}(x) = U(x)\mathcal{F}_{\mu\nu}(x)U^\dagger(x)$ transformation law directly from the definition $\mathcal{F}_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu\mathcal{A}_\nu - \partial_\mu\mathcal{A}_\nu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]$ and the non-abelian gauge transform of the $\mathcal{A}_\mu$ fields.

(e) Verify the Bianchi identity for the non-abelian tension fields $\mathcal{F}_{\mu\nu}(x)$:

$$D_\lambda\mathcal{F}_{\mu\nu} + D_\mu\mathcal{F}_{\nu\lambda} + D_\nu\mathcal{F}_{\lambda\mu} = 0. \tag{8}$$

Note the covariant derivatives in this equation.

Finally, consider the $SU(N)$ Yang–Mills theory — the non-abelian gauge theory that does not have any fields except $\mathcal{A}^a(x)$ and $\mathcal{F}^a(x)$; its Lagrangian is

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2g^2}\,\text{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) = \sum_a \tfrac{-1}{4}F^a_{\mu\nu}F^{a\mu\nu}. \tag{9}$$

(f) Show that the Euler–Lagrange field equations for the Yang–Mills theory can be written in covariant form as $D_\mu\mathcal{F}^{\mu\nu} = 0$.
Hint: first show that for an infinitesimal variation $\delta\mathcal{A}_\mu(x)$ of the non-abelian gauge fields, the tension fields vary according to $\delta\mathcal{F}_{\mu\nu}(x) = D_\mu\delta\mathcal{A}_\nu(x) - D_\nu\delta\mathcal{A}_\mu(x)$.

2. Continuing the previous problem, consider an $SU(N)$ gauge theory in which N^2-1 vector fields $A^a_\mu(x)$ interact with some "matter" fields $\phi_\alpha(x)$,

$$\mathcal{L} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) \;+\; \mathcal{L}_{\mathrm{mat}}(\phi, D_\mu\phi). \tag{10}$$

For the moment, let me keep the matter fields completely generic — they can be scalars, or vectors, or spinors, or whatever, and form any kind of a multiplet of the local $SU(N)$ symmetry as long as such multiplet is complete and non-trivial. All we need to know right now is that there are well-defined covariant derivatives $D_\mu\phi$ that depend on the gauge fields A^a_μ, which give rise to the currents

$$J^{a\mu} \;=\; -\frac{\partial\mathcal{L}_{\mathrm{mat}}}{\partial A^a_\mu}\,. \tag{11}$$

Collectively, these N^2-1 currents should form an adjoint multiplet $J^\mu = \sum_a(\frac{1}{2}\lambda^a)J^{a\mu}$ of the $SU(N)$ symmetry.

(a) Show that in this theory the equation of motion for the A^a_μ fields are $D_\mu F^{a\mu\nu} = J^{a\nu}$ and that consistency of these equations requires require the currents to be *covariantly conserved,*

$$D_\mu J^\mu \;=\; \partial_\mu J^\mu \;+\; i[\mathcal{A}_\mu, J^\mu] \;=\; 0, \tag{12}$$

or in components, $\partial_\mu J^{a\mu} - f^{abc}\mathcal{A}^b_\mu J^{c\mu} = 0$.

Note: a covariantly conserved current does *not* lead to a conserved charge, $(d/dt)\int d^3\mathbf{x}\, J^{a0}(\mathbf{x},t) \neq 0$!

Now consider a simple example of matter fields — a fundamental multiplet $\Psi(x)$ of N scalar fields $\Psi_i(x)$, with a Lagrangian

$$\mathcal{L}_{\mathrm{mat}} \;=\; D_\mu\Psi^\dagger D^\mu\Psi - m^2\Psi^\dagger\Psi - \frac{\lambda}{4}\big(\Psi^\dagger\Psi\big)^2, \qquad \mathcal{L}_{\mathrm{net}} \;=\; \mathcal{L}_{\mathrm{mat}} - \frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big). \tag{13}$$

(b) Derive the $SU(N)$ currents $J^{a\mu}$ for this set of fields and verify that under $SU(N)$ symmetries the currents transform covariantly into each other as members of the adjoint multiplet. That is, the $N\times N$ matrix $J^\mu = \sum_a(\frac{1}{2}\lambda^a)J^{a\mu}$ transforms according to eq. (3).

Hint: for any complex vectors Ψ and Ψ', $\sum_a(\Psi^\dagger\lambda^a\Psi')\lambda^a = 2\Psi'\otimes\Psi^\dagger - \frac{2}{N}(\Psi^\dagger\Psi')\times\mathbf{1}$.

(c) Finally, verify the covariant conservation $D_\mu J^{a\mu}$ of these currents when the scalar fields $\Psi_i(x)$ and $\Psi_i^\dagger(x)$ obey their equations of motion.

3. The last problem is about general multiplets of general gauge groups. Consider a Lie group G with generators $\hat{T}^a$ obeying commutation relations $[\hat{T}^a, \hat{T}^b] = if^{abc}\hat{T}^c$. Under an infinitesimal local symmetry

$$\mathcal{G}(x) \;=\; 1 \;+\; i\Lambda^a(x)\hat{T}^a \;+\; \cdots, \quad \text{infinitesimal } \Lambda^a(x), \tag{14}$$

the gauge fields $\mathcal{A}^a_\mu(x)$ transform as

$$\mathcal{A}^a_\mu(x) \;\to\; \mathcal{A}^a_\mu(x) \;-\; D_\mu\Lambda^a(x) \;=\; \mathcal{A}^a_\mu(x) \;-\; \partial_\mu\Lambda^a(x) \;-\; f^{abc}\Lambda^b(x)\mathcal{A}^c_\mu(x). \tag{15}$$

Other fields of the gauge theory (scalar, spinor, or whatever) must form complete multiplets of the gauge group G. In any such multiplet (m), the generators $\hat{T}^a$ are represented by $\mathrm{size}(m) \times \mathrm{size}(m)$ matrices $(T^a_{(m)})_\alpha^{\ \beta}$ satisfying similar commutation relations, $[T^a_{(m)}, T^b_{(m)}] = if^{abc}T^c_{(m)}$. The fields $\Psi_\alpha(x)$ belonging to such multiplet transform under infinitesimal gauge transforms (14) as

$$\Psi_\alpha(x) \;\to\; \Psi_\alpha(x) \;+\; i\Lambda^a(x)(T^a_{(m)})_\alpha^{\ \beta}\Psi_\beta(x) \tag{16}$$

and the covariant derivatives D_μ act on these fields as

$$D_\mu\Psi_\alpha(x) \;=\; \partial_\mu\Psi_\alpha(x) \;+\; i\mathcal{A}^a_\mu(x)(T^a_{(m)})_\alpha^{\ \beta}\Psi_\beta(x). \tag{17}$$

- Verify covariance of these derivatives under infinitesimal gauge transforms (14).

⋆ For extra challenge, only for the students familiar with the basic theory of Lie groups: Prove covariance of the derivatives (17) under finite gauge transforms.
Hint: use Lemma on the next page.

Lemma: For any finite symmetry $\mathcal{G} \in G$, the matrix $\big(R_{(m)}(\mathcal{G})\big)_{\alpha}^{\ \beta}$ representing this symmetry in the multiplet (m) satisfies

$$\big(R_{(m)}(\mathcal{G})\big)_{\alpha}^{\ \beta}\big(T^a_{(m)}\big)_{\beta}^{\ \gamma}\big(R^{-1}_{(m)}(\mathcal{G})\big)_{\gamma}^{\ \delta} \;=\; \big(T^b_{(m)}\big)_{\alpha}^{\ \delta} R^{ba}_{\rm adj}(\mathcal{G}) \tag{18}$$

where $R^{ba}_{\rm adj}(\mathcal{G})$ represents $\mathcal{G}$ in the adjoint multiplet. Note that the same $R^{ba}_{\rm adj}(\mathcal{G})$ appears on right hand sides of eqs. (18) for all multiplets (m) of G — and that's what allows us to use the same gauge fields $\mathcal{A}^a_\mu(x)$ to make covariant derivatives (17) for all multiplets of the gauge group G.