1. Consider an $O(N)$ symmetric Lagrangian for N interacting real scalar fields,

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^{N}(\partial_\mu \Phi_a)^2 - \frac{m^2}{2}\sum_{a=1}^{N}\Phi_a^2 - \frac{\lambda}{24}\left(\sum_{a=1}^{N}\Phi_a^2\right)^2. \tag{1}$$

By the Noether theorem, the continuous $SO(N)$ subgroup of (N) symmetry gives rise to $\frac{1}{2}N(N-1)$ conserved currents

$$J_{ab}^\mu(x) = -J_{ba}^\mu(x) = \Phi_a(x)\,\partial^\mu\Phi_b(x) - \Phi_b(x)\,\partial^\mu\Phi_a(x). \tag{2}$$

In the quantum field theory, these currents become operators

$$\begin{aligned}\hat{\mathbf{J}}_{ab}(\mathbf{x},t) &= -\hat{\mathbf{J}}_{ba}(\mathbf{x},t) = -\hat{\Phi}_a(\mathbf{x},t)\nabla\hat{\Phi}_b(\mathbf{x},t) + \hat{\Phi}_b(\mathbf{x},t)\nabla\hat{\Phi}_a(\mathbf{x},t),\\ \hat{J}_{ab}^0(\mathbf{x},t) &= -\hat{J}_{ba}^0(\mathbf{x},t)\ \hat{\Phi}_a(\mathbf{x},t)\hat{\Pi}_b(\mathbf{x},t) - \hat{\Phi}_b(\mathbf{x},t)\hat{\Pi}_a(\mathbf{x},t).\end{aligned} \tag{3}$$

This problem is about the net charge operators

$$\hat{Q}_{ab}(t) = -\hat{Q}_{ba}(t) = \int d^3\mathbf{x}\,\hat{J}^0(\mathbf{x}) = \int d^3\mathbf{x}\left(\hat{\Phi}_a(\mathbf{x},t)\hat{\Pi}_b(\mathbf{x},t) - \hat{\Phi}_b(\mathbf{x},t)\hat{\Pi}_a(\mathbf{x},t)\right). \tag{4}$$

(a) Write down the equal-time commutation relations for the quantum $\hat{\Phi}_a$ and $\hat{\Pi}_a$ fields. Also, write down the Hamiltonian operator for the interacting fields.

(b) Show that

$$\begin{aligned}\left[\hat{Q}_{ab}(t),\hat{\Phi}_c(\mathbf{x},\text{same } t)\right] &= i\delta_{bc}\hat{\Phi}_a(\mathbf{x},t) - i\delta_{ac}\hat{\Phi}_b(\mathbf{x},t),\\ \left[\hat{Q}_{ab}(t),\hat{\Pi}_c(\mathbf{x},\text{same } t)\right] &= i\delta_{bc}\hat{\Pi}_a(\mathbf{x},t) - i\delta_{ac}\hat{\Pi}_b(\mathbf{x},t),\end{aligned} \tag{5}$$

(c) Show that the all the $\hat{Q}_{ab}$ commute with the Hamiltonian operator $\hat{H}$. In the Heisenberg picture, this makes all the charge operators $\hat{Q}_{ab}$ time independent.

(d) Verify that the $\hat{Q}_{ab}$ obey commutation relations of the $SO(N)$ generators,

$$\left[\hat{Q}_{ab}, \hat{Q}_{cd}\right] \;=\; -i\delta_{[c[b}\hat{Q}_{a]d]} \;\equiv\; -i\delta_{bc}\hat{Q}_{ad} \;+\; i\delta_{ac}\hat{Q}_{bd} \;+\; i\delta_{bd}\hat{Q}_{ac} \;-\; i\delta_{ad}\hat{Q}_{bc}\,. \quad (6)$$

(e) In the Schrödinger picture $\hat{\Phi}_a(\mathbf{x})$ and $\hat{\Pi}_a(\mathbf{x})$ can be expanded into creation and annihilation operators as if they were free fields. Show that in terms of creation and annihilation operators, the charges (4) become

$$\hat{Q}_{ab} \;=\; \sum_{\mathbf{p}} \left(-i\hat{a}^{\dagger}_{\mathbf{p},a}\hat{a}_{\mathbf{p},b} \;+\; i\hat{a}^{\dagger}_{\mathbf{p},b}\hat{a}_{\mathbf{p},b} \right). \quad (7)$$

Finally, for $N = 2$ the $SO(2)$ symmetry is the phase symmetry of one complex field $\Phi = (\Phi_1 + i\Phi_2)/\sqrt{2}$ and its conjugate $\Phi^* = (\Phi_1 - i\Phi_2)/\sqrt{2}$. In the Fock space, they give rise to particles and anti-particles of opposite charges.

(f) Show that for $N = 2$

$$\hat{Q}_{21} \;=\; -\hat{Q}_{12} \;=\; \hat{N}_{\text{particles}} \;-\; \hat{N}_{\text{antiparticles}} \;=\; \sum_{\mathbf{p}} \left(\hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{p}} - \hat{b}^{\dagger}_{\mathbf{p}}\hat{b}_{\mathbf{p}} \right) \quad (8)$$

where

$$
\begin{aligned}
\hat{a}_{\mathbf{p}} &= \frac{\hat{a}_{\mathbf{p},1} + i\hat{a}_{\mathbf{p},2}}{\sqrt{2}} \quad \text{are particle annihilation operators,} \\
\hat{b}_{\mathbf{p}} &= \frac{\hat{a}_{\mathbf{p},1} - i\hat{a}_{\mathbf{p},2}}{\sqrt{2}} \quad \text{are antiparticle annihilation operators,} \\
\hat{a}^{\dagger}_{\mathbf{p}} &= \frac{\hat{a}^{\dagger}_{\mathbf{p},1} - i\hat{a}^{\dagger}_{\mathbf{p},2}}{\sqrt{2}} \quad \text{are particle creation operators,} \\
\hat{b}^{\dagger}_{\mathbf{p}} &= \frac{\hat{a}^{\dagger}_{\mathbf{p},1} + i\hat{a}^{\dagger}_{\mathbf{p},2}}{\sqrt{2}} \quad \text{are antiparticle creation operators.}
\end{aligned} \quad (9)
$$

2. Now consider a *massive* relativistic vector field $A^\mu(x)$ with the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} m^2 A_\mu A^\mu - A^\mu J_\mu \tag{10}$$

(in $\hbar = c = 1$ units) where the current $J^\mu(x)$ is a fixed source for the $A^\mu(x)$ field. Because of the mass term, the Lagrangian (10) is *not* gauge invariant. However, we *assume* that the current $J^\mu(x)$ is conserved, $\partial_\mu J^\mu(x) = 0$.

In an earlier homework (set 1, problem 1) we have derived the Euler–Lagrange equations for the massive vector field. In this problem, we develop the Hamiltonian formalism for the $A^\mu(x)$. Our first step is to identify the canonically conjugate "momentum" fields.

(a) Show that $\partial\mathcal{L}/\partial\dot{\mathbf{A}} = -\mathbf{E}$ but $\partial\mathcal{L}/\partial\dot{A}_0 \equiv 0$.

In other words, the canonically conjugate field to $\mathbf{A}(\mathbf{x})$ is $-\mathbf{E}(\mathbf{x})$ but the $A_0(\mathbf{x})$ does not have a canonical conjugate! Consequently,

$$H = \int d^3\mathbf{x} \left(-\dot{\mathbf{A}}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) - \mathcal{L} \right). \tag{11}$$

(b) Show that in terms of the $\mathbf{A}$, $\mathbf{E}$, and A_0 fields, and their *space* derivatives,

$$H = \int d^3\mathbf{x} \left\{ \tfrac{1}{2}\mathbf{E}^2 + A_0 \left(J_0 - \nabla \cdot \mathbf{E} \right) - \tfrac{1}{2} m^2 A_0^2 + \tfrac{1}{2} \left(\nabla \times \mathbf{A} \right)^2 + \tfrac{1}{2} m^2 \mathbf{A}^2 - \mathbf{J} \cdot \mathbf{A} \right\}. \tag{12}$$

Because the A_0 field does not have a canonical conjugate, the Hamiltonian formalism does not produce an equation for the time-dependence of this field. Instead, it gives us a time-independent equation relating the $A_0(\mathbf{x}, t)$ to the values of other fields *at the same time t*. Specifically, we have

$$\frac{\delta H}{\delta A_0(\mathbf{x})} \equiv \left.\frac{\partial\mathcal{H}}{\partial A_0}\right|_\mathbf{x} - \nabla \cdot \left.\frac{\partial\mathcal{H}}{\partial(\nabla A_0)}\right|_\mathbf{x} = 0. \tag{13}$$

At the same time, the vector fields $\mathbf{A}$ and $\mathbf{E}$ satisfy the Hamiltonian equations of motion,

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x}, t) &= -\left.\frac{\delta H}{\delta \mathbf{E}(\mathbf{x})}\right|_t \equiv -\left[\frac{\partial\mathcal{H}}{\partial\mathbf{E}} - \nabla_i\frac{\partial\mathcal{H}}{\partial(\nabla_i\mathbf{E})}\right]_{(\mathbf{x},t)}, \\
\frac{\partial}{\partial t}\mathbf{E}(\mathbf{x}, t) &= +\left.\frac{\delta H}{\delta \mathbf{A}(\mathbf{x})}\right|_t \equiv +\left[\frac{\partial\mathcal{H}}{\partial\mathbf{A}} - \nabla_i\frac{\partial\mathcal{H}}{\partial(\nabla_i\mathbf{A})}\right]_{(\mathbf{x},t)}.
\end{aligned}
\tag{14}
$$

(c) Write down the explicit form of all these equations.

(d) Verify that the equations you have just written down are equivalent to the relativistic Euler–Lagrange equations for the $A^\mu(x)$, namely

$$(\partial^\mu \partial_\mu + m^2)A^\nu = \partial^\nu(\partial_\mu A^\mu) + J^\nu \tag{15}$$

and hence $\partial_\mu A^\mu(x) = 0$ and $(\partial^\nu \partial_\nu + m^2)A^\mu = 0$ when $\partial_\mu J^\mu \equiv 0$, *cf.* homework #1.

3. Next, let's quantize the massive vector fields. Since classically the $-\mathbf{E}(\mathbf{x})$ fields are canonically conjugate momenta to the $\mathbf{A}(\mathbf{x})$ fields, the corresponding quantum fields $\hat{\mathbf{E}}(\mathbf{x})$ and $\hat{\mathbf{A}}(\mathbf{x})$ satisfy the canonical equal-time commutation relations

$$\begin{aligned}
[\hat{A}_i(\mathbf{x}, t), \hat{A}_j(\mathbf{y}, t)] &= 0, \\
[\hat{E}_i(\mathbf{x}, t), \hat{E}_j(\mathbf{y}, t)] &= 0, \\
[\hat{A}_i(\mathbf{x}, t), \hat{E}_j(\mathbf{y}, t)] &= -i\delta_{ij}\delta^{(3)}(\mathbf{x} - \mathbf{y})
\end{aligned} \tag{16}$$

(in the $\hbar = c = 1$ units). The currents also become quantum fields $\hat{J}^\mu(\mathbf{x}, t)$, but they are composed of some kind of charged degrees of freedom rather than the vector fields in question. Consequently, *at equal times* the currents $\hat{J}^\mu(\mathbf{x}, t)$ commute with both the $\hat{\mathbf{E}}(\mathbf{y}, t)$ and the $\hat{\mathbf{A}}(\mathbf{y}, t)$ fields.

The classical $A^0(\mathbf{x}, t)$ field does not have a canonical conjugate and its equation of motion does not involve time derivatives. In the quantum theory, $\hat{A}^0(\mathbf{x}, t)$ satisfies a similar time-independent constraint

$$m^2 \hat{A}^0(\mathbf{x}, t) = \hat{J}^0(\mathbf{x}, t) - \nabla \cdot \hat{\mathbf{E}}(\mathbf{x}, t), \tag{17}$$

but from the Hilbert space point of view this is an operatorial identity rather than an equation of motion. Consequently, the commutation relations of the scalar potential field follow from eqs. (16); in particular, at equal times the $\hat{A}^0(\mathbf{x}, t)$ commutes with the $\hat{\mathbf{E}}(\mathbf{y}, t)$ but does not commute with the $\hat{\mathbf{A}}(\mathbf{y}, t)$.

Finally, the Hamiltonian operator follows from the classical eq. (12), namely

$$
\begin{aligned}
\hat{H} &= \int d^3\mathbf{x} \left\{ \tfrac{1}{2}\hat{\mathbf{E}}^2 + \hat{A}_0\left(\hat{J}_0 - \nabla\cdot\hat{\mathbf{E}}\right) - \tfrac{1}{2}m^2\hat{A}_0^2 + \tfrac{1}{2}\left(\nabla\times\hat{\mathbf{A}}\right)^2 + \tfrac{1}{2}m^2\hat{\mathbf{A}}^2 - \hat{\mathbf{J}}\cdot\hat{\mathbf{A}} \right\} \\
&= \int d^3\mathbf{x} \left\{ \tfrac{1}{2}\hat{\mathbf{E}}^2 + \frac{1}{2m^2}\left(\hat{J}_0 - \nabla\cdot\hat{\mathbf{E}}\right)^2 + \tfrac{1}{2}\left(\nabla\times\hat{\mathbf{A}}\right)^2 + \tfrac{1}{2}m^2\hat{\mathbf{A}}^2 - \hat{\mathbf{J}}\cdot\hat{\mathbf{A}} \right\}
\end{aligned}
\tag{18}
$$

where the second line follows from the first and eq. (17).

Your task is to calculate the commutators $[\hat{A}_i(\mathbf{x},t),\hat{H}]$ and $[\hat{E}_i(\mathbf{x},t),\hat{H}]$ and write down the Heisenberg equations for the quantum vector fields. Make sure those equations are similar to the Hamilton equations for the classical fields.