1. First, an exercise in bosonic commutation relations

$$[\hat{a}_\alpha, \hat{a}_\beta] \;=\; 0, \qquad [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] \;=\; 0, \qquad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] \;=\; \delta_{\alpha\beta}\,. \tag{1}$$

   (a) Calculate the commutators $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma]$, $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta]$, $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta]$, and $[\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta, \hat{a}^\dagger_\mu \hat{a}_\nu]$.

   (b) For a single pair of $\hat{a}$ and $\hat{a}^\dagger$ operators, show that for any analytic function $f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$,

$$[\hat{a}, f(\hat{a}^\dagger)] \;=\; +f'(\hat{a}^\dagger) \quad\text{and}\quad [\hat{a}^\dagger, f(\hat{a})] \;=\; -f'(\hat{a}) \tag{2}$$

   where $f(\hat{a}) \stackrel{\text{def}}{=} f_0 + f_1\hat{a} + f_2(\hat{a})^2 + \cdots$ and likewise $f(\hat{a}^\dagger) \stackrel{\text{def}}{=} f_0 + f_1\hat{a}^\dagger + f_2(\hat{a}^\dagger)^2 + \cdots$.

   (c) Show that $e^{c\hat{a}}\hat{a}^\dagger e^{-c\hat{a}} = \hat{a}^\dagger + c$, $e^{c\hat{a}^\dagger}\hat{a}e^{-c\hat{a}^\dagger} = \hat{a} - c$, and hence for any analytic function f,

$$e^{c\hat{a}} f(\hat{a}^\dagger) e^{-c\hat{a}} \;=\; f(\hat{a}^\dagger + c) \quad\text{and}\quad e^{c\hat{a}^\dagger} f(\hat{a}) e^{-c\hat{a}^\dagger} \;=\; f(\hat{a} - c). \tag{3}$$

   (d) Now generalize (b) and (c) to any set of creation and annihilation operators $\hat{a}^\dagger_\alpha$ and $\hat{a}_\alpha$. Show that for any analytic function $f(\text{multiple } \hat{a}^\dagger_\alpha)$ of creation operators but not of the annihilation operators or a function $f(\text{multiple } \hat{a}_\alpha)$ of the annihilation operators but not of the creation operators,

$$\begin{aligned}
[\hat{a}_\alpha, f(\hat{a}^\dagger)] \;&=\; +\frac{\partial f(\hat{a}^\dagger)}{\partial \hat{a}^\dagger_\alpha}\,, \qquad [\hat{a}^\dagger_\alpha, f(\hat{a})] \;=\; -\frac{\partial f(\hat{a})}{\partial \hat{a}_\alpha}\,, \\
\exp\left(\sum\nolimits_\alpha c_\alpha \hat{a}_\alpha\right) f(\hat{a}^\dagger) \exp\left(-\sum\nolimits_\alpha c_\alpha \hat{a}_\alpha\right) \;&=\; f(\text{each } \hat{a}^\dagger_\alpha \to \hat{a}^\dagger_\alpha + c_\alpha), \\
\exp\left(\sum\nolimits_\alpha c_\alpha \hat{a}^\dagger_\alpha\right) f(\hat{a}) \exp\left(-\sum\nolimits_\alpha c_\alpha \hat{a}^\dagger_\alpha\right) \;&=\; f(\text{each } \hat{a}_\alpha \to \hat{a}_\alpha - c_\alpha).
\end{aligned} \tag{4}$$

2. An operator acting on identical bosons can be described in terms of N–particle wave functions (the *first-quantized* formalism) or in terms of creation and annihilation operators in the Fock space (the *second-quantized* formalism). This exercise is about converting the operators from one formalism to another.

The keys to this conversion are single-particle wave functions $\phi_\alpha(\mathbf{x})$ of states $|\alpha\rangle$ and *symmetrized* N-particle states wave functions

$$
\begin{aligned}
\phi_{\alpha\beta\cdots\omega}(\mathbf{x}_1, \mathbf{x}_2 \ldots, \mathbf{x}_N) &= \frac{1}{\sqrt{D}} \sum_{(\tilde{\alpha},\tilde{\beta},\ldots,\tilde{\omega})}^{\substack{\text{distinct permutations}\\ \text{of } (\alpha,\beta,\ldots,\omega)}} \phi_{\tilde{\alpha}}(\mathbf{x}_1) \times \phi_{\tilde{\beta}}(\mathbf{x}_2) \times \cdots \times \phi_{\tilde{\omega}}(\mathbf{x}_N) \\
&= \frac{1}{T\sqrt{D}} \sum_{(\tilde{\alpha},\tilde{\beta},\ldots,\tilde{\omega})}^{\substack{\text{all permutations}\\ \text{of } (\alpha,\beta,\ldots,\omega)}} \phi_{\tilde{\alpha}}(\mathbf{x}_1) \times \phi_{\tilde{\beta}}(\mathbf{x}_2) \times \cdots \times \phi_{\tilde{\omega}}(\mathbf{x}_N)
\end{aligned}
\tag{5}
$$

of N-boson states $|\alpha, \beta, \ldots, \omega\rangle$. In eqs. (5), D is the number of *distinct* permutations of single-particle states $(\alpha, \beta, \ldots, \omega)$ and T is the number of trivial permutations. (A trivial permutation permutes states that happen to be the same, a distinct permutation permutes different states only.) In terms of the occupation numbers n_γ,

$$
T = \prod_\gamma n_\gamma!\,, \qquad D = \frac{N!}{T}\,. \tag{6}
$$

(a) Consider a generic N-particle quantum state $|N;\psi\rangle$ with a totally symmetric wave-function $\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N)$. Show that the $(N+1)$–particle state $|N+1, \psi'\rangle = \hat{a}^\dagger_\alpha\,|N;\psi\rangle$ has wave function

$$
\psi'(\mathbf{x}_1, \ldots, \mathbf{x}_{N+1}) = \frac{1}{\sqrt{N+1}} \sum_{i=1}^{N+1} \phi_\alpha(\mathbf{x}_i) \times \psi(\mathbf{x}_1, \ldots, \cancel{\mathbf{x}_i}, \ldots, \mathbf{x}_{N+1}). \tag{7}
$$

Hint: First prove this for wave-functions of the form (5). Then use the fact that states $|\alpha_1, \ldots, \alpha_N\rangle$ form a complete basis of the N-boson Hilbert space.

(b) Show that the $(N-1)$–particle state $|N-1,\psi''\rangle = \hat{a}_\alpha\,|N;\psi\rangle$ has wave-function

$$\psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;=\; \sqrt{N}\int\! d^3\mathbf{x}_N\,\phi^*_\alpha(\mathbf{x}_N)\times\psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N). \tag{8}$$

Hint: the $\hat{a}_\alpha$ is the hermitian conjugate of the $\hat{a}^\dagger_\alpha$, so for any $|N-1,\widetilde{\psi}\rangle$,
$\langle N-1,\widetilde{\psi}|\,\hat{a}_\alpha\,|N,\psi\rangle = \langle N,\psi|\,\hat{a}^\dagger_\alpha\,|N-1,\widetilde{\psi}\rangle^*$.

Next, consider one-body operators, *i.e.* additive operators acting on one particle at a time. In the first-quantized formalism they act on N–particle states according to

$$\hat{A}^{(1)}_{\rm net} \;=\; \sum_{i=1}^{N}\hat{A}_1(i^{\rm th}\ \text{particle}) \tag{9}$$

where $\hat{A}_1$ is some kind of a one-particle operator (such as momentum $\hat{\mathbf{p}}$, or kinetic energy $\frac{1}{2m}\hat{\mathbf{p}}^2$, or potential $V(\hat{\mathbf{x}})$, *etc.*, *etc.*). In the second-quantized formalism such operators become

$$\hat{A}^{(2)}_{\rm net} \;=\; \sum_{\alpha,\beta}\langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\hat{a}^\dagger_\alpha\hat{a}_\beta\,. \tag{10}$$

(c) Verify that the two operators have the same matrix elements between any two N-boson states $|N,\psi\rangle$ and $|N,\widetilde{\psi}\rangle$, $\langle N,\widetilde{\psi}|\,\hat{A}^{(1)}_{\rm net}\,|N,\psi\rangle = \langle N,\widetilde{\psi}|\,\hat{A}^{(2)}_{\rm net}\,|N,\psi\rangle$.
Hint: use $\hat{A}_1 = \sum_{\alpha,\beta}|\alpha\rangle\,\langle\alpha|\,\hat{A}_1\,|\beta\rangle\,\langle\beta|$.

(d) Now let $\hat{A}^{(2)}_{\rm net}$, $\hat{B}^{(2)}_{\rm net}$, and $\hat{C}^{(2)}_{\rm net}$ be three second-quantized net one-body operators corresponding to the single-particle operators $\hat{A}_1$, $\hat{B}_1$, and $\hat{C}_1$. Show that if $\hat{C}_1 = [\hat{A}_1,\hat{B}_1]$ then $\hat{C}^{(2)}_{\rm net} = \left[\hat{A}^{(2)}_{\rm net},\hat{B}^{(2)}_{\rm net}\right]$.

Finally, consider two-body operators, *i.e.* additive operators acting on two particles at a time. Given a two-particle operator $\hat{B}_2$ — such as $V(\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2)$ — the *net* B operator acts in the first-quantized formalism according to

$$\hat{B}^{(1)}_{\rm net} \;=\; \tfrac{1}{2}\sum_{i\neq j}\hat{B}_2(i^{\rm th}\ \text{and}\ j^{\rm th}\ \text{particles}), \tag{11}$$

and in the second-quantized formalism according to

$$\hat{B}^{(2)}_{\rm net} \;=\; \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta}\left(\left(\langle\alpha|\otimes\langle\beta|\right)\hat{B}_2\left(|\gamma\rangle\otimes|\delta\rangle\right)\right)\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta\,. \tag{12}$$

(e) Again, show these two operators have the same matrix elements between any two N-boson states, $\langle N,\widetilde{\psi}|\,\hat{A}^{(1)}_{\rm net}\,|N,\psi\rangle = \langle N,\widetilde{\psi}|\,\hat{A}^{(2)}_{\rm net}\,|N,\psi\rangle$ for any $\langle N,\widetilde{\psi}|$ and $|N,\psi\rangle$.

(f) Now let $\hat{A}_1$ be a one-particle operator, let $\hat{B}_2$ and $\hat{C}_2$ be two-body operators, and let $\hat{A}^{(2)}_{\rm net}$, $\hat{B}^{(2)}_{\rm net}$, and $\hat{C}^{(2)}_{\rm net}$ be the corresponding second-quantized operators according to eqs. (10) and (12).

Show that if $\hat{C}_2 = \left[\left(\hat{A}_1(1^{\rm st}) + \hat{A}_1(2^{\rm nd})\right), \hat{B}_2\right]$ then $\hat{C}^{(2)}_{\rm net} = \left[\hat{A}^{(2)}_{\rm net}, \hat{B}^{(2)}_{\rm net}\right]$.

3. The rest of this homework is about coherent states of harmonic oscillators and free quantum fields. Let us start with a harmonic oscillator with Hamiltonian $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}$.

(a) For any complex number ξ we define a *coherent state* $|\xi\rangle \stackrel{\rm def}{=} \exp\bigl(\xi\hat{a}^\dagger - \xi^*\hat{a}\bigr)\,|0\rangle$. Show that

$$|\xi\rangle \;=\; e^{-|\xi|^2/2}\,e^{\xi\hat{a}^\dagger}\,|0\rangle \quad \text{and} \quad \hat{a}\,|\xi\rangle \;=\; \xi\,|\xi\rangle\,. \tag{13}$$

(b) Use $\hat{a}\,|\xi\rangle = \xi\,|\xi\rangle$ to show that the (coordinate-space) wave function of a coherent state $|\xi\rangle$ is a Gaussian wave packet of the same width as the ground state $|0\rangle$. Also, show that the central position $\bar{x}$ and the central momentum $\bar{p}$ of this packet are related to the real and the imaginary parts of ξ,

$$\bar{x} \;=\; \sqrt{\frac{2\hbar}{\omega m}} \times {\rm Re}\,\xi, \quad \bar{p} \;=\; \sqrt{2\hbar\omega m} \times {\rm Im}\,\xi, \quad \xi \;=\; \frac{m\omega\bar{x} + i\bar{p}}{\sqrt{2m\omega\hbar}}\,. \tag{14}$$

(c) Use $\hat{a}\,|\xi\rangle = \xi\,|\xi\rangle$ and $\langle\xi|\,\hat{a}^\dagger = \xi^*\,\langle\xi|$ to calculate the uncertainties Δx and Δp in a coherent state and verify their minimality: $\Delta x\Delta p = \frac{1}{2}\hbar$. Also, verify $\delta n = \sqrt{\bar{n}}$ where $\bar{n} \stackrel{\rm def}{=} \langle\hat{n}\rangle = |\xi|^2$.

The coherent states are not stationary, they evolve with time but stay coherent — a coherent state $|\xi_0\rangle$ at time $t = 0$ becomes $|\xi(t)\rangle$ at later times — while the central position $\bar{x}$ and $\bar{p}$ of the wave packet move according to the classical equations of motion for the harmonic oscillator.

(d) Check that such classical motion calls for $\xi(t) = \xi_0 \times e^{-i\omega t}$, then check that the corresponding coherent state $|\xi(t)\rangle$ obeys the time-dependent Schrödinger equation $i\hbar\frac{d}{dt}|\xi(t)\rangle = \hat{H}\,|\xi(t)\rangle$.

(e) The coherent states are not quite orthogonal to each other.
Calculate their probability overlaps $|\langle\eta|\xi\rangle|^2$.

Now consider the coherent states of multi-oscillator systems such as quantum fields. In particular, let us focus on the creation and annihilation fields $\hat{\Psi}^\dagger(\mathbf{x})$ and $\hat{\Psi}(\mathbf{x})$ for non-relativistic spinless bosons.

(f) Generalize (a) and construct coherent states $|\Phi\rangle$ which satisfy

$$\hat{\Psi}(\mathbf{x})\,|\Phi\rangle \;=\; \Phi(\mathbf{x})\,|\Phi\rangle \tag{15}$$

for any given classical complex field $\Phi(\mathbf{x})$.

(g) Show that for any such coherent state, $\Delta N = \sqrt{\bar{N}}$ where

$$\bar{N} \stackrel{\text{def}}{=} \langle\Phi|\,\hat{N}\,|\Phi\rangle \;=\; \int d\mathbf{x}\,|\Phi(\mathbf{x})|^2. \tag{16}$$

(h) Let the Hamiltonian for the quantum non-relativistic fields be

$$\hat{H} \;=\; \int d\mathbf{x}\left(\frac{\hbar^2}{2M}\nabla\hat{\Psi}^\dagger(\mathbf{x})\cdot\nabla\hat{\Psi}(\mathbf{x}) \;+\; V(\mathbf{x})\times\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}(\mathbf{x})\right). \tag{17}$$

Show that for any classical field configuration $\Phi(\mathbf{x},t)$ obeying the classical field equation

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{x},t) \;=\; \left(-\frac{\hbar^2}{2M}\nabla^2 + V(\mathbf{x})\right)\Phi(\mathbf{x},t), \tag{18}$$

the time-dependent coherent state $|\Phi\rangle\,(t)$ satisfies the true Schrödinger equation

$$i\hbar\frac{d}{dt}\,|\Phi\rangle \;=\; \hat{H}\,|\Phi\rangle\,. \tag{19}$$

(i) Finally, show that the quantum overlap $|\langle\Phi_1|\Phi_2\rangle|^2$ between two different coherent states is exponentially small for any *macroscopic* difference $\delta\Phi(\mathbf{x}) = \Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})$ between the two field configurations.