1. When an *exact* symmetry of a quantum field theory is spontaneously broken down, it gives rise to exactly massless Goldstone bosons. But when the spontaneously broken symmetry was only approximate to begin with, the would-be Goldstone bosons are no longer exactly massless but only relatively light. The best-known examples of such pseudo-Goldstone bosons are the pi-mesons $\pi^\pm$ and π^0, which are indeed much lighter then other hadrons. The Quantum ChromoDynamics theory (QCD) of strong interactions has an approximate chiral isospin symmetry $SU(2)_L \times SU(2)_R \cong \mathrm{Spin}(4)$. This symmetry would be exact if the two lightest quark flavors u and d were massless; in real life, the masses m_u and m_d are small but non quite zero, and the symmetry is only approximate. Somehow (and people are still arguing how), the chiral isospin symmetry is spontaneously broken down to the ordinary isospin symmetry $SU(2) \cong \mathrm{Spin}(3)$, and the 3 generators of the broken $\mathrm{Spin}(4)/\mathrm{Spin}(3)$ give rise to 3 (pseudo) Goldstone bosons $\pi^\pm$ and π^0.

   QCD is a rather complicated theory, so it is often convenient to describe the physics of the spontaneously broken chiral symmetry in terms of a simpler theory with similar symmetries. For example, the *linear sigma model* is a theory of 4 real scalar fields, an isosinglet $\sigma(x)$ and an isotriplet $\underset{\sim}{\pi}(x)$ comprising $\pi^1(x)$, $\pi^2(x)$ and $\pi^3(x)$ (or equivalently, $\pi^0(x) \equiv \pi^3(x)$ and $\pi^\pm(x) \equiv \big(\pi^1(x) \pm i\pi^2(x)\big)/\sqrt{2}$). The Lagrangian

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial_\mu\sigma)^2 \;+\; \tfrac{1}{2}(\partial_\mu\underset{\sim}{\pi})^2 \;-\; \frac{\lambda}{8}\left(\sigma^2 + \underset{\sim}{\pi}^2 - f^2\right)^2 \;+\; \beta\lambda f^2 \times \sigma \tag{1}$$

   is invariant under the $SO(4)$ rotations of the four fields, except for the last term which we treat as a perturbation. In class we saw that for $\beta = 0$ this theory has $SO(4)$ spontaneously broken to $SO(3)$ and hence 3 massless Goldstone bosons — the pions. In this exercise, we let $\beta > 0$ but $\beta \ll f$ to show how this leads to pions being massive but light.

   (a) Show that the scalar potential of the linear sigma model with $\beta > 0$ has a unique minimum at

$$\langle\underset{\sim}{\pi}\rangle \;=\; 0 \quad\text{and}\quad \langle\sigma\rangle \;=\; f \;+\; \beta \;+\; O(\beta^2/f). \tag{2}$$

   (b) Expand the fields around this minimum and show that the pions are light while the σ particle is much heavier. Specifically, $M_\pi^2 \approx \lambda f\beta$ while $M_\sigma^2 \approx \lambda f(f+\beta) \approx \lambda f^2 \gg M_\pi^2$.

2. In this problem the spontaneously broken symmetry is exact but more complicated. Consider an $N \times N$ matrix $\Phi(x)$ of complex scalar fields $\Phi^i{}_j(x)$, $i, j = 1, \ldots, N$. In matrix notations, the Lagrangian is

$$\mathcal{L} \;=\; \operatorname{tr}\left(\partial^\mu \Phi^\dagger\, \partial_\mu \Phi\right) \;-\; V(\Phi^\dagger \Phi) \tag{3}$$

where the potential is

$$V \;=\; \frac{\alpha}{2}\operatorname{tr}\left(\Phi^\dagger\Phi\Phi^\dagger\Phi\right) \;+\; \frac{\beta}{2}\left(\operatorname{tr}\left(\Phi^\dagger\Phi\right)\right)^2 \;+\; m^2 \operatorname{tr}\left(\Phi^\dagger\Phi\right). \tag{4}$$

(a) Show that this theory has global symmetry group $G = SU(N)_L \times SU(N)_R \times U(1)$ acting as

$$\Phi(x) \;\to\; e^{i\theta} U_L \Phi(x) U_R^\dagger, \qquad U_L, U_R \in SU(N). \tag{5}$$

($\star$) *Optional exercise, only for experts in group theory:*
Show that the theory has no other continuous symmetries besides G and Poincare (Lorentz and translations of spacetime).

From now on, we take $\alpha, \beta > 0$ but $m^2 < 0$. In this regime, V is minimized for non-zero vacuum expectation values $\langle \Phi \rangle \neq 0$ of the scalar fields.

(b) Let $(\kappa_1, \ldots, \kappa_N)$ be eigenvalues of the hermitian matrix $\Phi^\dagger\Phi$. Express the potential (4) in terms of these eigenvalues and show that the minimum lies at

$$\kappa_1 \;=\; \kappa_2 \;=\; \cdots \;=\; \kappa_N \;=\; C^2 \;=\; \frac{-m^2}{\alpha + N\beta} \;>\; 0. \tag{6}$$

In terms of the matrix Φ, eq. (6) means $\Phi = C \times$ a unitary matrix. All such minima are related by symmetries (5) to $\Psi = C \times$ the unit matrix, so without loss of generality we may assume that the vacuum lies at

$$\langle \Phi \rangle \;=\; C \times \mathbf{1}_{N\times N} \quad i.e. \quad \left\langle \Phi^i{}_j \right\rangle \;=\; C \times \delta^i_j\,. \tag{7}$$

(c) Show that in this vacuum, the symmetry group of the theory is spontaneously broken down to $SU(N)$; in terms of eq. (5), the unbroken symmetries have $U_L = U_R \in SU(N)$ and $\theta = 0$.

Let's expand the theory around the vacuum (7). For convenience, let's also decompose the complex matrix Φ into its hermitian and anti-hermitian parts,

$$\Phi(x) \;=\; C\times\mathbf{1}_{N\times N} \;+\; \frac{\varphi_1(x)\,+\,i\,\varphi_2(x)}{\sqrt{2}} \qquad \text{where } \varphi_1^\dagger\equiv\varphi_1 \text{ and } \varphi_2^\dagger\equiv\varphi_2\,. \tag{8}$$

(d) Expand the Lagrangian in powers of φ_1 and φ_2 and use the quadratic part $\mathcal{L}_2$ to determine the particle spectrum of the theory.

(e) Check the quantum numbers of the massless particles and verify that they agree with the Nambu–Goldstone theorem for the spontaneously broken symmetries of the theory.

3. Now let's gauge the $SU(N)_L\times SU(N)_R\times U(1)$ symmetry of the previous problem. Naturally, this requires abelian gauge fields $B_\mu(x)$ and non-abelian matrix-valued gauge fields $L_\mu(x)$ and $R_\mu(x)$; in components, $L_\mu(x)=\sum_a\frac12\lambda^a\times L_\mu^a(x)$ and $R_\mu(x)=\sum_a\frac12\lambda^a\times R_\mu^a(x)$ where $a=1,\ldots,(N^2-1)$ and λ^a are the Gell–Mann matrices of $SU(N)$. The Lagrangian now is

$$\mathcal{L} \;=\; -\tfrac14 B_{\mu\nu}B^{\mu\nu} \;-\; \tfrac12\operatorname{tr}\left(L_{\mu\nu}L^{\mu\nu}\right) \;-\; \tfrac12\operatorname{tr}\left(R_{\mu\nu}R^{\mu\nu}\right) \;+\; \operatorname{tr}\Big(D^\mu\Phi^\dagger\,D_\mu\Phi\Big) \;-\; V(\Phi^\dagger\Phi) \tag{9}$$

where the scalar potential V is as in eq. (4), and

$$\begin{aligned}
B_{\mu\nu} \;&=\; \partial_\mu B_\nu \;-\; \partial_\nu B_\mu\,,\\
L_{\mu\nu} \;&=\; \partial_\mu L_\nu \;-\; \partial_\nu L_\mu \;+\; ig[L_\mu,L_\nu],\\
R_{\mu\nu} \;&=\; \partial_\mu R_\nu \;-\; \partial_\nu R_\mu \;+\; ig[R_\mu,R_\nu],\\
D_\mu\Phi \;&=\; \partial_\mu\Phi \;+\; ig'B_\mu\Phi \;+\; igL_\mu\Phi \;-\; ig\Phi R_\mu\,,\\
D_\mu\Phi^\dagger \;&=\; (D_\mu\Phi)^\dagger \;=\; \partial_\mu\Phi^\dagger \;-\; ig'B_\mu\Phi^\dagger \;+\; igR_\mu\Phi^\dagger \;-\; ig\Phi^\dagger L_\mu\,.
\end{aligned} \tag{10}$$

For simplicity, I assume equal gauge couplings $g_L=g_R=g$ for the two $SU(N)$ factors of the gauge group, but the abelian coupling g' is different.

As in the previous problem, we take $\alpha,\beta>0$ but $m^2<0$ so the scalar's vacuum expectation values $\langle\Phi\rangle$ are as in eq. (7), and the $SU(N)_L\times SU(N)_R\times U(1)$ gauge symmetry is broken down to $SU(N)$.

(a) Write down the mass matrix for the vector fields. Show that B_μ and $X^a_\mu = \frac{1}{\sqrt{2}}(L^a_\mu - R^a_\mu)$ vectors become massive while $V^a_\mu = \frac{1}{\sqrt{2}}(L^a_\mu + R^a_\mu)$ remain massless.

(b) Find the effective Lagrangian for the massless vector fields $V^a_\mu(x)$ by freezing all the other fields, *i.e.* setting $\Phi(x) \equiv \langle\Phi\rangle$, $B_\mu(x) \equiv 0$ and $X^a_\mu(x) \equiv 0$. Show that this Lagrangian describes a Yang–Mills theory with gauge group $SU(N)_V$ and gauge coupling $g_V = g/\sqrt{2}$.

(c) In the unitary gauge for the broken gauge symmetries the $\Phi(x)$ scalar field matrix is hermitian, $\Phi^\dagger(x) \equiv \Phi(x)$, or in terms of eq. (8) $\phi_2(x) \equiv 0$. To show that this is a good gauge condition, show that: (1) it fixes all the broken gauge symmetries but does not fix the un-broken symmetries, and (2) any $\Phi(x)$ is gauge-equivalent to a hermitian $\Phi'(x)$ via a non-singular gauge transform.

(d) Finally, rewrite the whole Lagrangian (9) in terms of fields of definite mass — V_μ, X_μ, B_μ and $\delta\Psi$ — and their derivatives that are covariant with respect to the unbroken $SU(N)_V$. For simplicity, fix the unitary gauge.