1. Quantum mechanics of a fixed number of *relativistic* particles does not work (except as an approximation) because of problems with relativistic causality. Indeed, consider a single free relativistic spinless particle with Hamiltonian

$$\hat{H} \;=\; +\sqrt{M^2 + \hat{\mathbf{P}}^2} \tag{1}$$

(in the $c = \hbar = 1$ units). In the coordinate picture, this Hamiltonian is a horrible integro-differential operator, but that's only a technical problem. The real problem concerns the time evolution kernel

$$U(\mathbf{x} - \mathbf{y}; t) \;=\; \langle \mathbf{x}, t | \mathbf{y}, t_0 = 0 \rangle^{\text{Heisenberg}}_{\text{picture}} \;=\; \langle \mathbf{x} | \exp(-it\hat{H}) | \mathbf{y} \rangle^{\text{Schroedinger}}_{\text{picture}} . \tag{2}$$

(a) Show that

$$U(\mathbf{x}-\mathbf{y}; t) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3} \exp\bigl(i\mathbf{k}(\mathbf{y}-\mathbf{x}) - i\omega(\mathbf{k})t\bigr) \;=\; \frac{-i}{4\pi^2 r} \int\limits_{-\infty}^{+\infty} dk\, k\, \exp\bigl(irk - it\omega(k)\bigr) \tag{3}$$

where $\omega(k) = \sqrt{M^2 + k^2}$ and $r = |\mathbf{x} - \mathbf{y}|$.

(b) Take the limit $t \to +\infty$, $r \to \infty$ while the ratio r/t stays fixed. Specifically, let $(r/t) < 1$ so we stay inside the future light cone.

Show that in this limit, the evolution kernel becomes

$$U(\mathbf{x} - \mathbf{y}; t) \;\approx\; \frac{(-iM)^{3/2}}{4\pi^{3/2}} \, \frac{t}{(t^2 - r^2)^{5/4}} \times \exp(-iM\sqrt{t^2 - r^2}). \tag{4}$$

Hint: Use the *saddle point method* to evaluate the integral (3). If you are not familiar with this method — or with any of the related methods for approximating integrals of the form $\int dx\, f(x) \times \exp\bigl(-Ag(x)\bigr)$ for $A \to \infty$ — read my notes on the saddle-point method. Those notes were originally written for a QM class so they include the Airy function example; you do not need that example, just the method itself.

(c) Finally, take a similar limit but go outside the light cone, thus fixed $(r/t) > 1$ while $r, t \to +\infty$. Show that in this limit, the kernel becomes

$$U(\mathbf{x}-\mathbf{y};t) \approx \frac{iM^{3/2}}{4\pi^{3/2}} \frac{t}{(r^2-t^2)^{5/4}} \times \exp(-M\sqrt{r^2-t^2}). \tag{5}$$

Hint: again, use the saddle point method.

Eq. (5) shows that the kernel diminishes exponentially outside the light cone, *but it does not vanish!* Thus, given a particle localized at point $\mathbf{y}$ at the time $t_0 = 0$, after time $t > 0$, its wave function is *mostly* limited to the future light cone $r < t$, *but there is an exponential tail outside the light cone.* In other words, the probability of superluminal motion is exponentially small but non-zero.

Obviously, such superluminal propagation cannot be allowed in a consistently relativistic theory. And that's why relativistic quantum mechanics of a single particle is inconsistent. Likewise, relativistic quantum mechanics of any fixed number of particles does not work, except as an approximation.

In the quantum field theory, this paradox is resolved by allowing for creation and annihilation of particles. Quantum field operators acting at points x and y outside each others' lightcones can either create a particle at x and then annihilate it at y, or else annihilate it at y and then create it at x. I will show in class that the two effects *precisely* cancel each other, so altogether there is no propagation outside the light cone. That's how relativistic QFT is perfectly causal while the relativistic QM is not.

2. The second problem is about the quantum massive vector field $A_\mu(x)$ and its expansion into creation and annihilation operators. The massive vector field has appeared in two previous homeworks: in set#1 you've derived its equation of motion from the Lagrangian, while in set#4 you've developed the Hamiltonian formalism and quantized the field. For the present exercise you will need the equal-times commutation relations of the quantum fields,

$$[\hat{A}^i(\mathbf{x}), \hat{A}^j(\mathbf{y})] = 0, \quad [\hat{E}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})] = 0, \quad [\hat{A}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})] = -i\delta^{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{6}$$

(in $\hbar = 1, c = 1$ units), the Hamiltonian operator

$$\hat{H} = \int d^3\mathbf{x} \left(\tfrac{1}{2}\hat{\mathbf{E}}^2 + \frac{(\nabla\cdot\hat{\mathbf{E}})^2}{2m^2} + \tfrac{1}{2}(\nabla\times\hat{\mathbf{A}})^2 + \tfrac{1}{2}m^2\hat{\mathbf{A}}^2 \right). \tag{7}$$

for the free fields (*i.e.*, for $\hat{J}^\mu(\mathbf{x}) \equiv 0$), and the operatorial identity

$$\hat{A}^0(x) = -\frac{\nabla\cdot\hat{\mathbf{E}}(x)}{m^2} \tag{8}$$

(again, for $\hat{J}^0(\mathbf{x}) \equiv 0$).

In general, a QFT has a creation operator $\hat{a}^\dagger_{\mathbf{k},\lambda}$ and an annihilation operator $\hat{a}_{\mathbf{k},\lambda}$ for each plane wave with momentum $\mathbf{k}$ and polarization λ. The massive vector fields have 3 independent polarizations corresponding to 3 orthogonal unit 3–vectors. One may use any basis of 3 such vectors $\mathbf{e}_\lambda(\mathbf{k})$, and it's often convenient to make them $\mathbf{k}$–dependent and complex; in the complex case, orthogonality+unit length mean

$$\mathbf{e}_\lambda(\mathbf{k}) \cdot \mathbf{e}^*_{\lambda'}(\mathbf{k}) = \delta_{\lambda,\lambda'}\,. \tag{9}$$

Of particular convenience is the helicity basis of eigenvectors of the vector product $i\mathbf{k}\times$, namely

$$i\mathbf{k}\times\mathbf{e}_\lambda(\mathbf{k}) = \lambda|\mathbf{k}|\mathbf{e}_\lambda(\mathbf{k}), \qquad \lambda = -1, 0, +1. \tag{10}$$

By convention, the phases of the complex helicity eigenvectors are chosen such that

$$\mathbf{e}_0(\mathbf{k}) = \frac{\mathbf{k}}{|\mathbf{k}|}\,, \quad \mathbf{e}^*_{\pm1}(\mathbf{k}) = -\mathbf{e}_{\mp1}(\mathbf{k}), \quad \mathbf{e}_\lambda(-\mathbf{k}) = -\mathbf{e}^*_\lambda(+\mathbf{k}). \tag{11}$$

As a first step towards constructing the $\hat{a}_{\mathbf{k},\lambda}$ and $\hat{a}^\dagger_{\mathbf{k},\lambda}$ operators, we Fourier transform the vector fields $\hat{\mathbf{A}}(\mathbf{x})$ and $\hat{\mathbf{E}}(\mathbf{x})$ and then decompose the vectors $\hat{\mathbf{A}}_\mathbf{k}$ and $\hat{\mathbf{E}}_\mathbf{k}$ into helicity components,

$$
\begin{aligned}
\hat{\mathbf{A}}(\mathbf{x}) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_\lambda e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_\lambda(\mathbf{k})\, \hat{A}_{\mathbf{k},\lambda}\,, \qquad & \hat{A}_{\mathbf{k},\lambda} &= \int d^3\mathbf{x}\, e^{-i\mathbf{k}\mathbf{x}} \mathbf{e}^*_\lambda(\mathbf{k}) \cdot \hat{\mathbf{A}}(\mathbf{x}), \\
\hat{\mathbf{E}}(\mathbf{x}) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_\lambda e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_\lambda(\mathbf{k})\, \hat{E}_{\mathbf{k},\lambda}\,, \qquad & \hat{E}_{\mathbf{k},\lambda} &= \int d^3\mathbf{x}\, e^{-i\mathbf{k}\mathbf{x}} \mathbf{e}^*_\lambda(\mathbf{k}) \cdot \hat{\mathbf{E}}(\mathbf{x}).
\end{aligned}
\tag{12}
$$

(a) Show that $\hat{A}^\dagger_{\mathbf{k},\lambda} = -\hat{A}_{-\mathbf{k},\lambda}$, $\hat{E}^\dagger_{\mathbf{k},\lambda} = -\hat{E}_{-\mathbf{k},\lambda}$, and derive the equal-time commutation relations for the $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$ operators.

(b) Show that

$$
\hat{H} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_\lambda \left(\frac{C_{\mathbf{k},\lambda}}{2} \hat{E}^\dagger_{\mathbf{k},\lambda} \hat{E}_{\mathbf{k},\lambda} + \frac{\omega^2_{\mathbf{k}}}{2C_{\mathbf{k},\lambda}} \hat{A}^\dagger_{\mathbf{k},\lambda} \hat{A}_{\mathbf{k},\lambda} \right) \tag{13}
$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$ and $C_{\mathbf{k},\lambda} = 1 + \delta_{\lambda,0}(\mathbf{k}^2/m^2)$.

(c) Define creation and annihilation operators according to

$$
\hat{a}_{\mathbf{k},\lambda} = \frac{\omega_{\mathbf{k}} \hat{A}_{\mathbf{k},\lambda} - iC_{\mathbf{k},\lambda} \hat{E}_{\mathbf{k},\lambda}}{\sqrt{C_{\mathbf{k},\lambda}}}\,, \qquad \hat{a}^\dagger_{\mathbf{k},\lambda} = \frac{\omega_{\mathbf{k}} \hat{A}^\dagger_{\mathbf{k},\lambda} + iC_{\mathbf{k},\lambda} \hat{E}^\dagger_{\mathbf{k},\lambda}}{\sqrt{C_{\mathbf{k},\lambda}}}\,, \tag{14}
$$

and verify that they satisfy equal-time bosonic commutation relations (relativistically normalized).

(d) Show that

$$
\hat{H} = \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2\omega_{\mathbf{k}}} \sum_\lambda \omega_{\mathbf{k}}\, \hat{a}^\dagger_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} + \text{const.} \tag{15}
$$

(e) Next, consider the time dependence of the free vector field in the Heisenberg picture. Show that

$$
\hat{\mathbf{A}}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2\omega_{\mathbf{k}}} \sum_\lambda \sqrt{C_{\mathbf{k},\lambda}} \left(e^{-ikx} \mathbf{e}_\lambda(\mathbf{k})\, \hat{a}_{\mathbf{k},\lambda}(0) + e^{+ikx} \mathbf{e}^*_\lambda(\mathbf{k})\, \hat{a}^\dagger_{\mathbf{k},\lambda}(0) \right)_{k^0 = +\omega_{\mathbf{k}}} . \tag{16}
$$

(f) Use eq. (8) to write down a similar formula for the $\hat{A}^0(\mathbf{x},t)$. (use eq. (8)).

(g) Combine the results of parts (e) and (f) into a relativistic formula for the 4–vector field $\hat{A}^\mu(x)$, namely

$$\hat{A}_\mu(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\sum_\lambda \left(e^{-ikx} f_\mu(\mathbf{k},\lambda)\,\hat{a}_{\mathbf{k},\lambda}(0) \;+\; e^{+ikx} f^*_\mu(\mathbf{k},\lambda)\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\right)_{k^0=+\omega_\mathbf{k}} \tag{17}$$

where the 4–vectors $f^\mu(\mathbf{k},\lambda)$ obtain by Lorentz boosting of purely-spatial polarization vectors $\mathbf{e}_\lambda(\mathbf{k})$ into the moving particle's frame. Specifically,

$$f^\mu(\mathbf{k},\lambda) \;=\; \begin{cases} \big(0, \mathbf{e}_\lambda(\mathbf{k})\big) & \text{for } \lambda = \pm 1, \\ \left(\frac{|\mathbf{k}|}{m}, \frac{\omega_\mathbf{k}}{m}\frac{\mathbf{k}}{|\mathbf{k}|}\right) & \text{for } \lambda = 0, \end{cases} \tag{18}$$

and they satisfy

$$k_\mu f^\mu_{\mathbf{k},\lambda} \;=\; 0, \qquad f^\mu_{\mathbf{k},\lambda}\left(f^*_{\mathbf{k},\lambda'}\right)_\mu \;=\; -\delta_{\lambda,\lambda'}\,. \tag{19}$$

(h) Finally, verify that the quantum vector field (17) satisfies the free equations of motion $\partial_\mu \hat{A}^\mu(x) = 0$ and $(\partial^2 + m^2)\hat{A}^\mu(x) = 0$; moreover, each mode in the expansion (17) satisfies the equations of motions without any help from the other modes.

3. The last problem concerns the Feynman propagator for the massive vector field. I recommend you do this problem after I explain the scalar field's Feynman propagator in class on Thursday 10/18 and Tuesday 10/23. Meanwhile, start working on the problems **1** and **2** of this set!

(a) First, a lemma: Show that

$$\sum_\lambda f^\mu(\mathbf{k},\lambda) f^{\nu *}(\mathbf{k},\lambda) \;=\; -g^{\mu\nu} \;+\; \frac{k^\mu k^\nu}{m^2}\,. \tag{20}$$

(b) Next, calculate the "vacuum sandwich" of two vector fields and show that

$$\begin{aligned} \langle 0|\,\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \;&=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\left[\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right) e^{-ik(x-y)}\right]_{k^0=+\omega_\mathbf{k}} \\ &=\; \left(-g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right) D(x-y). \end{aligned} \tag{21}$$

(c) And now, the Feynman propagator: Show that

$$
\begin{aligned}
G_F^{\mu\nu} \;\equiv\; \langle 0|\,\mathbf{T}^*\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \;&=\; \left(-g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right) G_F^{\rm scalar}(x-y) \\
&= \int\!\frac{d^4\mathbf{k}}{(2\pi)^4}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right)\frac{ie^{-ik(x-y)}}{k^2-m^2+i0}
\end{aligned}
\tag{22}
$$

where

$$
\mathbf{T}^*\hat{A}^\mu(x)\hat{A}^\nu(y) \;=\; \mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y) \;+\; \frac{i}{m^2}\,\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y),
\tag{23}
$$

is the *modified* time-ordered product of the vector fields. The purpose of this modification[⋆] is to absorb the $\delta^{(4)}(x-y)$ stemming from the $\partial_0\partial_0 G_F(x-y)$.

(d) Finally, write the classical action for the free vector field as

$$
S \;=\; \tfrac{1}{2}\int\!d^4x\,A_\mu(x)\,\mathcal{D}^{\mu\nu}A_\nu(x)
\tag{24}
$$

where $\mathcal{D}^{\mu\nu}$ is a differential operator and show that the Feynman propagator (22) is a Green's function of this operator,

$$
\mathcal{D}_x^{\mu\nu}G^F_{\nu\lambda} \;=\; +i\delta^\mu_\lambda\delta^{(4)}(x-y).
\tag{25}
$$

⋆ See *Quantum Field Theory* by Claude Itzykson and Jean–Bernard Zuber.