1. First, an exercise in Dirac matrices γ^μ. Please do not assume any specific form of these 4×4 matrices, just use the anti-commutation relations

$$\gamma^\mu\gamma^\nu \;+\; \gamma^\nu\gamma^\mu \;=\; 2g^{\mu\nu} \tag{1}$$

and $(\gamma^0)^\dagger = +\gamma^0$ while $(\gamma^i)^\dagger = -\gamma^i$ for $i = 1,2,3$.

(a) Show that $\gamma^\alpha\gamma_\alpha = 4$, $\gamma^\alpha\gamma^\nu\gamma_\alpha = -2\gamma^\nu$, $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = 4g^{\mu\nu}$, and $\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha = -2\gamma^\nu\gamma^\mu\gamma^\lambda$. Hint: use $\gamma^\alpha\gamma^\nu = 2g^{\nu\alpha} - \gamma^\nu\gamma^\alpha$ repeatedly.

(b) The electron field in the EM background obeys the *covariant Dirac equation* $\bigl(i\gamma^\mu D_\mu \,-\, m\bigr)\Psi(x) \,=\, 0$ where $D_\mu\Psi = \partial_\mu\Psi - ieA_\mu\Psi$. Show that this equation implies

$$\bigl(D_\mu D^\mu \,+\, m^2 \,+\, qF_{\mu\nu}S^{\mu\nu}\bigr)\,\Psi(x) \;=\; 0. \tag{2}$$

Besides the 4 Dirac matrices γ^μ, there is another useful matrix $\gamma^5 \stackrel{\rm def}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$.

(c) Show that γ^5 anticommutes with each of the γ^μ matrices — $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ — and commutes with all the spin matrices $S^{\mu\nu}$.

(d) Show that γ^5 is hermitian and that $(\gamma^5)^2 = 1$.

(e) Show that $\gamma^5 = (i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ and $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = -24i\epsilon^{\kappa\lambda\mu\nu}\,\gamma^5$.

(f) Show that $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = -6i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5$.

(g) Show that any 4×4 matrix Γ is a unique linear combination of the following 16 matrices: 1, γ^μ, $\frac{1}{2}\gamma^{[\mu}\gamma^{\nu]} = -2iS^{\mu\nu}$, $\gamma^5\gamma^\mu$, and γ^5.

∗ My conventions here are: $\epsilon^{0123} = -1$, $\epsilon_{0123} = +1$, $\gamma^{[\mu}\gamma^{\nu]} = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$, $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\lambda\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\lambda$, *etc.*

Now consider Dirac matrices in spacetime dimensions $d \neq 4$. Such matrices always satisfy the Clifford algebra (1), but their sizes depend on d.

Let $\Gamma = i^n\gamma^0\gamma^1\cdots\gamma^{d-1}$ be the generalization of the γ^5 to d dimensions; the pre-factor $i^n = \pm i$ or ± 1 is chosen such that $\Gamma = \Gamma^\dagger$ and $\Gamma^2 = +1$.

(h) *For even* d, Γ anticommutes with all the γ^μ. Prove this, and use this fact to show that there are 2^d independent products of the γ^μ matrices, and consequently the matrices should be $2^{d/2} \times 2^{d/2}$.

(i) *For odd* d, Γ commutes with all the Γ^μ — prove this. Consequently, one can set $\Gamma = +1$ or $\Gamma = -1$; the two choices lead to in-equivalent sets of the γ^μ.

Classify the independent products of the γ^μ for odd d and show that their net number is 2^{d-1}; consequently, the matrices should be $2^{(d-1)/2} \times 2^{(d-1)/2}$.

2. Now let's go back to $d = 3+1$ and learn about the Weyl spinors and Weyl spinor fields. Since all the spin matrices $S^{\mu\nu}$ commute with the γ^5, for all *continuous* Lorentz symmetries $L^\mu_{\ \nu}$ their Dirac-spinor representations $M_D(L) = \exp\bigl(-\frac{i}{2}\Theta_{\alpha\beta}S^{\alpha\beta}\bigr)$ are block-diagonal in the eigenbasis of the γ^5. This makes the Dirac spinor Ψ a *reducible* multiplet of the continuous Lorentz group $SO^+(3,1)$ — it comprises two different irreducible 2-component spinor multiplets called the left-handed Weyl spinor ψ_L and the right-handed Weyl spinor ψ_R.

This decomposition becomes clear in the Weyl convention for the Dirac matrices where

$$\gamma^\mu = \begin{pmatrix} 0 & \bar\sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \quad \text{where} \quad \begin{aligned} \sigma^\mu &\stackrel{\text{def}}{=} \bigl(\mathbf{1}_{2\times 2}, -\boldsymbol{\sigma}\bigr), \\ \bar\sigma^\mu &\stackrel{\text{def}}{=} \bigl(\mathbf{1}_{2\times 2}, +\boldsymbol{\sigma}\bigr), \end{aligned} \tag{3}$$

and consequently

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \implies M_D(L) = \begin{pmatrix} M_L(L) & 0 \\ 0 & M_R(L) \end{pmatrix}. \tag{4}$$

(a) Check that the γ^5 matrix indeed has this form and write down explicit matrices for the $S^{\mu\nu}$ in the Weyl convention.

In the Weyl convention

$$\Psi_{\text{Dirac}}(x) = \begin{pmatrix} \psi_L(x), \\ \psi_R(x) \end{pmatrix} \quad \text{where} \quad \begin{aligned} \psi'_L(x') &= M_L(L)\psi_L(x), \\ \psi'_R(x') &= M_R(L)\psi_R(x). \end{aligned} \tag{5}$$

(b) Express the Dirac Lagrangian $\overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$ in terms of the Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$ (and their conjugates $\psi_L^\dagger(x)$ and $\psi_R^\dagger(x)$) and show that for $m = 0$ the two Weyl spinor fields become independent from each other.

Now let's work out the explicit forms of the M_L and M_R matrices for the pure rotations and for the pure boosts.

(c) Show that for a space rotation R through angle ϕ around axis $\mathbf{n}$,

$$M_L(R) \;=\; M_R(R) \;=\; \exp\bigl(-\tfrac{i}{2}\phi\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr) \in SU(2), \tag{6}$$

so both the left-handed and the right-handed Weyl spinors ψ_L and ψ_R transform as the ordinary 2-component spinors of the Spin(3) rotation group.

(d) Show that for a Lorentz boost of *rapidity* r in the direction $\mathbf{n}$

$$M_L(B) \;=\; \exp(-\tfrac{1}{2}\,r\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr) \quad \text{while} \quad M_R(B) \;=\; \exp(+\tfrac{1}{2}\,r\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr). \tag{7}$$

The rapidity is related to the β and γ parameters of a Lorentz boost as $\beta = \tanh(r)$, $\gamma = \cosh(r)$. For two successive boosts in the same directions, the rapidities add up, $r_{1+2} = r_1 + r_2$. In terms of the β and γ parameters,

$$M_L \;=\; \sqrt{\gamma}\times\sqrt{1\,-\,\beta\,\mathbf{n}\cdot\boldsymbol{\sigma}}\,, \qquad M_R \;=\; \sqrt{\gamma}\times\sqrt{1\,+\,\beta\,\mathbf{n}\cdot\boldsymbol{\sigma}}\,. \tag{8}$$

Note that for the boosts $M_L \neq M_R$ — so the left-handed Weyl spinors transform differently from the right-handed Weyl spinors — and that both M_L and M_R are non-unitary.

The LH Weyl spinor and the RH Weyl spinor are two different, *in-equivalent* multiplets of the continuous Lorentz symmetry $\mathrm{SO}^+(3,1)$, or rather $\mathrm{Spin}(3,1)$. However, each of them is equivalent to the complex conjugate of the other spinor multiplet.

(e) Show that for any $L \in \mathrm{SO}^+(3,1)$,

$$M_R \;=\; \sigma_2 M_L^* \sigma_2 \quad \text{and} \quad M_L \;=\; \sigma_2 M_R^* \sigma_2\,, \tag{9}$$

and consequently $\sigma_2\psi_R^*$ transforms like ψ_L while $\sigma_2\psi_L^*$ transforms like ψ_R,

$$\sigma_2\psi_R'^*(x') \;=\; M_L\,\sigma_2\psi_R^*(x) \quad \text{and} \quad \sigma_2\psi_L'^*(x') \;=\; M_R\,\sigma_2\psi_L^*(x). \tag{10}$$

3. The third problem is about the plane-wave solutions $e^{-ipx}u(p,s)$ and $e^{+ipx}v(p,x)$ of the Dirac equation. In all these waves $p^0 = +E_{\mathbf{p}} = +\sqrt{\mathbf{p}^2 + m^2}$ while the 4–component spinors $u(p,s)$ and $v(p,s)$ satisfy

$$(\not{p} - m)u(p,s) \;=\; 0, \quad (\not{p} + m)v(p,s) \;=\; 0 \tag{11}$$

and are normalized to

$$u^\dagger(p,s)u(p,s') \;=\; v^\dagger(p,s)v(p,s') \;=\; 2E\delta_{s,s'}\,. \tag{12}$$

Let's writing down explicit formulae for these spinors in the Weyl basis for the γ^μ matrices.

(a) Show that for $\mathbf{p} = 0$,

$$u(\mathbf{p} = \mathbf{0}, s) \;=\; \begin{pmatrix} \sqrt{m}\,\xi_s \\ \sqrt{m}\,\xi_s \end{pmatrix} \tag{13}$$

where ξ_s is a two-component $SO(3)$ spinor encoding the electron's spin state. The ξ_s are normalized to $\xi_s^\dagger \xi_{s'} = \delta_{s,s'}$.

(b) For other momenta, $u(p,s) = M(\text{boost})u(\mathbf{p} = 0, s)$ for the boost that turns $(m, \vec{0})$ into p^μ. Use eqs. (8) to show that

$$u(p,s) \;=\; \begin{pmatrix} \sqrt{E - \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E + \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix} \;=\; \begin{pmatrix} \sqrt{p_\mu\bar{\sigma}^\mu}\,\xi_s \\ \sqrt{p_\mu\sigma^\mu}\,\xi_s \end{pmatrix}. \tag{14}$$

(c) Use similar arguments to show that

$$v(p,s) \;=\; \begin{pmatrix} +\sqrt{E - \mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E + \mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix} \;=\; \begin{pmatrix} +\sqrt{p_\mu\bar{\sigma}^\mu}\,\eta_s \\ -\sqrt{p_\mu\sigma^\mu}\,\eta_s \end{pmatrix} \tag{15}$$

where η_s are two-component $SO(3)$ spinors normalized to $\eta_s^\dagger \eta_{s'} = \delta_{s,s'}$.

Physically, the η_s should have opposite spins from ξ_s — the holes in the Dirac sea have opposite spins (as well as p^μ) from the missing negative-energy particles. Mathematically, this requires $\eta_s^\dagger \mathbf{S}\eta_s = -\xi_s^\dagger \mathbf{S}\xi_s$; we may solve this condition by letting $\eta_s = \sigma_2\xi_s^* = \pm i\xi_{-s}$.

(d) Check that this is a solution, then show that it leads to $v(p,s) = \gamma^2 u^*(p,s)$.

(e) Show that for ultra-relativistic electrons or positrons of definite helicity $\lambda = \pm\frac{1}{2}$, the Dirac plane waves become *chiral* — *i.e.*, dominated by one of the two irreducible Weyl spinor components $\psi_L(x)$ or $\psi_R(x)$ of the Dirac spinor $\Psi(x)$ while the other component becomes negligible. Specifically,

$$\begin{aligned} u(p,-\tfrac{1}{2}) &\approx \sqrt{2E}\begin{pmatrix}\xi_L\\ 0\end{pmatrix}, \qquad & u(p,+\tfrac{1}{2}) &\approx \sqrt{2E}\begin{pmatrix}0\\ \xi_R\end{pmatrix}, \\ v(p,-\tfrac{1}{2}) &\approx -\sqrt{2E}\begin{pmatrix}0\\ \eta_L\end{pmatrix}, \qquad & v(p,+\tfrac{1}{2}) &\approx \sqrt{2E}\begin{pmatrix}\eta_R\\ 0\end{pmatrix}. \end{aligned} \tag{16}$$

Note that for the electrons the helicity and the chirality are both left or both right, but for the positrons the chirality is opposite from the helicity.

Back in problem 2(b) we saw that for $m = 0$ the LH and the RH Weyl spinor fields decouple from each other. Now this exercise show us which particle modes comprise each Weyl spinor: *The $\psi_L(x)$ contains the left-handed fermions and the right-handed antifermions, while the $\psi_R(x)$ contains the right-handed fermions and the left-handed antifermions.*

Finally, let's establish some basis-independent properties of the Dirac spinors $u(p,s)$ and $v(p,s)$ — although you may use the Weyl basis to verify them.

(f) Show that

$$\bar{u}(p,s)u(p,s') = +2m\delta_{s,s'}\,, \quad \bar{v}(p,s)v(p,s') = -2m\delta_{s,s'}\,; \tag{17}$$

note that the normalization here is different from eq. (12) for the $u^\dagger u$ and $v^\dagger v$.

(g) There are only two independent $SO(3)$ spinors, hence $\sum_s \xi_s\xi_s^\dagger = \sum_s \eta_s^\dagger\eta_s = \mathbf{1}_{2\times 2}$. Use this fact to show that

$$\sum_{s=1,2} u_\alpha(p,s)\bar{u}_\beta(p,s) = (\not{p}+m)_{\alpha\beta} \quad \text{and} \quad \sum_{s=1,2} v_\alpha(p,s)\bar{v}_\beta(p,s) = (\not{p}-m)_{\alpha\beta}\,. \tag{18}$$

4. Next, a little exercise on Lorentz algebra. Show that a *massless* particle state $|p,\lambda\rangle$ of definite momentum and helicity satisfies the eigenstate-like condition

$$\epsilon_{\alpha\lambda\mu\nu}\hat{J}^{\lambda\mu}\hat{P}^{\nu}\,|p,\lambda\rangle \;=\; 2\lambda\hat{P}_{\lambda}\,|p,\lambda\rangle\,. \tag{19}$$

⋆ Finally, *for extra challenge* show that the continuous Lorentz group — or rather its double cover Spin(3,1) — is isomorphic to $SL(2,\mathbf{C})$, the group of complex 2×2 matrices (unitary or not) with det $=1$. The correspondence between the Lorentz symmetries $L^{\mu}_{\ \nu}$ and the $SL(2,\mathbf{C})$ matrices M works like this: One one hand, $M=M_L(L)$, the LH Weyl spinor representation of L; on the other hand we may reconstruct L from M according to

$$V'^{\mu} \;=\; L^{\mu}_{\ \nu}V^{\nu} \quad\Longleftrightarrow\quad V'_{\mu}\bar{\sigma}^{\mu} \;=\; M\big(V_{\mu}\bar{\sigma}^{\mu}\big)M^{\dagger}. \tag{20}$$

To make this work, show that:

(a) For any L, $\det(M_L(L))=1$ so that $M_L(L)\in SL(2,\mathbf{C})$.

(b) For any $M\in SL(2,\mathbf{C})$ eq. (20) defined a Lorentz symmetry.
Hint: prove and use $\det(V_{\mu}\bar{\sigma}^{\mu})=V_{\mu}V^{\mu}$.

(c) For a unitary M eq. (20) leads to L being a pure rotation of space for which $M_L=M$, *cf.* eq. (6).

(d) For an hermitian M eq. (20) leads to L being a pure Lorentz boost for which $M_L=M$, *cf.* eq. (8).

(e) Use polar decomposition $M=\text{unitary}\times\text{hermitian}$ to show that any $M\in SL(2,\mathbf{C})$ leads to a continuous Lorentz transform $L=R\times B$.