1. The *parity* **P** is the im-proper Lorentz symmetry that reflects the space but not the time, $(\mathbf{x}, t) \to (-\mathbf{x}, +t)$. This symmetry acts on Dirac spinor fields according to

$$\widehat{\Psi}'(-\mathbf{x}, +t) \;=\; \pm\gamma^0\widehat{\Psi}(+\mathbf{x}, +t) \tag{1}$$

where the overall $\pm$ sign is the *intrinsic parity* of the fermion species.

(a) Verify that the Dirac equation transforms covariantly under (1) and that the Dirac Lagrangian is invariant (apart from $\mathcal{L}(\mathbf{x}, t) \to \mathcal{L}(-\mathbf{x}, t)$).

In the Fock space, eq. (1) becomes

$$\widehat{\mathbf{P}}\widehat{\Psi}(\mathbf{x}, t)\widehat{\mathbf{P}} \;=\; \pm\gamma^0\widehat{\Psi}(-\mathbf{x}, t) \tag{2}$$

for some unitary operator $\widehat{\mathbf{P}}$ that squares to one. Let's find how this operator acts on the particles and their states.

(b) Check that the plane-wave solutions $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ from the previous homework — set 8, problem 3 — satisfy $u(-\mathbf{p}, s) = +\gamma^0 u(\mathbf{p}, s)$ and $v(-\mathbf{p}, s) = -\gamma^0 v(\mathbf{p}, s)$, then use these relations to show that eq. (2) implies

$$\begin{aligned} \widehat{\mathbf{P}}\,\hat{a}_{\mathbf{p},s}\,\widehat{\mathbf{P}} \;&=\; \pm\hat{a}_{-\mathbf{p},+s}\,, \quad & \widehat{\mathbf{P}}\,\hat{a}^\dagger_{\mathbf{p},s}\,\widehat{\mathbf{P}} \;&=\; \pm\hat{a}^\dagger_{-\mathbf{p},+s}\,, \\ \widehat{\mathbf{P}}\,\hat{b}_{\mathbf{p},s}\,\widehat{\mathbf{P}} \;&=\; \mp\hat{b}_{-\mathbf{p},+s}\,, \quad & \widehat{\mathbf{P}}\,\hat{b}^\dagger_{\mathbf{p},s}\,\widehat{\mathbf{P}} \;&=\; \mp\hat{b}^\dagger_{-\mathbf{p},+s}\,, \end{aligned} \tag{3}$$

and hence

$$\widehat{\mathbf{P}}\,|F(\mathbf{p}, s)\rangle \;=\; \pm\,|F(-\mathbf{p}, +s)\rangle \quad \text{and} \quad \widehat{\mathbf{P}}\,\big|\overline{F}(\mathbf{p}, s)\big\rangle \;=\; \mp\,\big|\overline{F}(-\mathbf{p}, +s)\big\rangle\,. \tag{4}$$

Note that the fermion and the antifermion have opposite intrinsic parities!

Now consider the bilinear products of a Dirac field $\Psi(x)$ and its conjugate $\overline{\Psi}(x)$. Generally, such products have form $\overline{\Psi}\Gamma\Psi$ where Γ is one of 16 matrices discussed in the previous homework — set 8, problem 1. Altogether, we have

$$S = \overline{\Psi}\Psi, \quad V^\mu = \overline{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \overline{\Psi}\tfrac{i}{2}\gamma^{[\mu}\gamma^{\nu]}\Psi, \quad A^\mu = \overline{\Psi}\gamma^5\gamma^\mu\Psi, \quad \text{and} \quad P = \overline{\Psi}i\gamma^5\Psi. \tag{5}$$

(c) Show that all the bilinears (5) are Hermitian.
Hint: despite the Fermi statistics, $\left(\Psi^\dagger_\alpha\Psi_\beta\right)^\dagger = +\Psi^\dagger_\beta\Psi_\alpha$; use this fact to show that $\left(\overline{\Psi}\Gamma\Psi\right)^\dagger = \overline{\Psi\Gamma}\Psi$.

(d) Show that under *continuous* Lorentz symmetries, the S and the P transform as scalars, the V^μ and the A^μ as vectors, and the $T^{\mu\nu}$ as an antisymmetric tensor.

(e) Find the transformation rules of the bilinears (5) under parity and show that while S is a true scalar and V is a true (polar) vector, P is a pseudoscalar and A is an axial vector.

2. The charge conjugation symmetry $\mathbf{C}$ does not transform the space or the time; instead, it exchanges particles with antiparticles, for example the electrons e^- with the positrons e^+,

$$\widehat{\mathbf{C}}\left|e^-(\mathbf{p},s)\right\rangle = \left|e^+(\mathbf{p},s)\right\rangle, \quad \widehat{\mathbf{C}}\left|e^+(\mathbf{p},s)\right\rangle = \left|e^-(\mathbf{p},s)\right\rangle. \tag{6}$$

In class I have explained that in the fermionic Fock space $\mathbf{C}$ is realized as a unitary operator $\widehat{\mathbf{C}} = \widehat{\mathbf{C}}^\dagger = \widehat{\mathbf{C}}^{-1}$ which acts on the creation and annihilation operators according to

$$\widehat{\mathbf{C}}\,\hat{a}^\dagger_{\mathbf{p},s}\widehat{\mathbf{C}} = \pm\hat{b}^\dagger_{\mathbf{p},s}, \quad \widehat{\mathbf{C}}\,\hat{b}^\dagger_{\mathbf{p},s}\widehat{\mathbf{C}} = \pm\hat{a}^\dagger_{\mathbf{p},s}, \quad \widehat{\mathbf{C}}\,\hat{a}_{\mathbf{p},s}\widehat{\mathbf{C}} = \pm\hat{b}_{\mathbf{p},s}, \quad \widehat{\mathbf{C}}\,\hat{b}_{\mathbf{p},s}\widehat{\mathbf{C}} = \pm\hat{a}_{\mathbf{p},s}, \tag{7}$$

where the overall $\pm$ sign is the intrinsic C-parity which depends on the fermionic species. I have also showed that eqs. (7) imply that the quantum Dirac fields $\widehat{\Psi}(x)$ and $\widehat{\overline{\Psi}}(x)$

transform under charge conjugation to

$$\widehat{\mathbf{C}}\widehat{\Psi}(x)\widehat{\mathbf{C}} \;=\; \pm\gamma^2\widehat{\Psi}^*(x) \quad\text{and}\quad \widehat{\mathbf{C}}\widehat{\overline{\Psi}}(x)\widehat{\mathbf{C}} \;=\; \pm\widehat{\overline{\Psi}}^*(x)\gamma^2. \tag{8}$$

(a) Show that that the *classical* Dirac Lagrangian is invariant under the charge conjugation up to a total spacetime derivative. Note that in the classical limit the Dirac fields *anticommute* with each other, $\Psi^*_\alpha\Psi_\beta = -\Psi_\beta\Psi^*_\alpha$. Also, similar to the hermitian conjugation of quantum fields, the complex conjugation of classical fermionic fields reverses their order, $(F_1F_2)^* = F_2^*F_1^* = -F_1^*F_2^*$.

Next, consider the charge-conjugation properties of the Dirac bilinears (5). To avoid the operator-ordering problems, take the classical limit where $\Psi(x)$ and $\Psi^\dagger(x)$ *anticommute* with each other, $\Psi_\alpha\Psi^\dagger_\beta = -\Psi^\dagger_\beta\Psi_\alpha$.

(b) Show that $\mathbf{C}$ turns $\overline{\Psi}\Gamma\Psi$ into $\overline{\Psi}\Gamma^c\Psi$ where $\Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2$.

(c) Calculate Γ^c for all 16 independent matrices Γ and find out which Dirac bilinears are C–even and which are C–odd.

3. Now consider a neutral bound state of a charged Dirac fermion F and the corresponding antifermion, for example a $q\bar{q}$ meson or a positronium "atom" (a hydrogen-atom-like bound state of e^- and e^+). In the Fock space, such a bound state can be constructed as

$$|B(\mathbf{p}_{\text{tot}} = 0)\rangle \;=\; \int\!\frac{d^3\mathbf{p}_{\text{red}}}{(2\pi)^3}\sum_{s_1,s_2}\psi(\mathbf{p}_{\text{red}}, s_1, s_2)\times\hat{a}^\dagger(+\mathbf{p}_{\text{red}}, s_1)\,\hat{b}^\dagger(-\mathbf{p}_{\text{red}}, s_2)\,|0\rangle \tag{9}$$

for some wave-function ψ of the reduced momentum and the two spins.

(a) Suppose this bound state has a definite orbital angular momentum L and definite net spin S. Show that the intrinsic C-parity and the P-parity of this bound state are

$$C \;=\; (-1)^{L+S}, \quad P \;=\; (-1)^{L+1}. \tag{10}$$

(b) Use eqs. (10) to explain why the annihilation rate of the ground 1S state of the positronium "atom" depends on the net spin: the $S = 0$ state decays much faster than the $S = 1$ state. Note: since the EM fields couple linearly to the electric charges and currents (which are reversed by $\widehat{\mathbf{C}}$), each photon has $C = -1$.

4. In the last homework — set 8, problem 2 — we saw that a left-handed Weyl spinor ψ_L is equivalent to the complex conjugate of a right-handed Weyl spinor ψ_R and vice verse. Consequently, a Dirac spinor field $\Psi(x)$ together with its conjugate $\overline{\Psi}(x)$ are equivalent to two left-handed Weyl spinor fields $\chi(x)$ and $\tilde{\chi}(x)$ together with their right-handed conjugates $\sigma_2\chi^*(x)$ and $\sigma^2\tilde{\chi}^*(x)$. In the Weyl basis (where γ^5 is diagonal)

$$\Psi(x) \;=\; \begin{pmatrix} \chi(x) \\ -\sigma_2\tilde{\chi}^*(x) \end{pmatrix}, \qquad \overline{\Psi}(x) \;=\; \Big(-\tilde{\chi}^\top(x)\sigma_2\,,\, \chi^\dagger(x)\Big). \tag{11}$$

(a) Show that up to a total derivative,

$$\mathcal{L}_{\text{Dirac}} \;\equiv\; \overline{\Psi}(i\,\partial\!\!\!/ - m)\Psi \;=\; i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi \;+\; i\tilde{\chi}^\dagger\bar{\sigma}^\mu\partial_\mu\tilde{\chi} \;+\; m\chi^\top\sigma_2\,\tilde{\chi} \;+\; m\chi^\dagger\sigma_2\,\tilde{\chi}^*. \tag{12}$$

Hint: $\sigma_2\sigma^\mu\sigma_2 = (\bar{\sigma}^\mu)^* = (\bar{\sigma}^\mu)^\top$.
Note the $\chi \leftrightarrow \tilde{\chi}$ symmetry of the Lagrangian (12): In the last two terms, the σ^2 matrix is antisymmetric but the fields are fermionic, hence $\chi^\top\sigma_2\tilde{\chi} = -\tilde{\chi}^\top\sigma_2^\top\chi = +\tilde{\chi}^\top\sigma_2\chi$ and likewise $\chi^\dagger\sigma_2\tilde{\chi}^* = +\tilde{\chi}^\dagger\sigma_2\chi^*$.

(b) Express the Dirac bi-linears (5) in terms of the Weyl spinors χ and $\tilde{\chi}$ (and their hermitian conjugates). For simplicity, assume classical anticommuting fermionic fields.

(c) Work out how the parity **P**, the charge conjugation **C**, and the combined **CP** symmetry act on the Weyl spinor fields $\chi(x)$ and $\tilde{\chi}(x)$.

Now let's generalize from two Weyl spinor fields comprising a Dirac field Ψ to any number N of left-handed Weyl spinor fields $\chi_j(x)$ with free Lagrangian

$$\mathcal{L} \;=\; \sum_j i\chi^{j\dagger}\bar{\sigma}^\mu\partial_\mu\chi_j \;+\; \tfrac{1}{2}\sum_{j,k} M^{jk}\chi_j^\top\sigma_2\,\chi_k \;+\; \tfrac{1}{2}\sum_{j,k} M^*_{jk}\chi^{j\dagger}\sigma_2\,\chi^{k*}. \tag{13}$$

The mass matrix M^{jk} here must be symmetric, $M^{jk} = M^{kj}$, but it may be complex rather than real.

(d) Show that the Weyl equations for the χ_j fields lead to Klein–Gordon equations

$$\partial^2 \chi_i \;+\; (M^* M)_i^{\;j} \chi_j \;=\; 0, \tag{14}$$

which mean that the physical fermion masses^2 are eigenvalues of the $M^*M = M^\dagger M$ matrix.
Hints: use $\sigma_2 (\sigma^\mu)^\top \sigma_2 = \bar\sigma^\mu$ and $\sigma^\mu \bar\sigma^\nu + \sigma^\nu \bar\sigma^\mu = 2g^{\mu\nu}$.

Now consider the combined **CP** symmetry of the Weyl fermions. The simplest realization of this symmetry acts similarly on all the spinors,

$$\mathbf{CP} : \chi_j(\mathbf{x}, t) \;\to =\; \pm i \times \sigma_2 \chi_j^*(-\mathbf{x}, +t), \quad \text{same } \pm i \; \forall j. \tag{15}$$

Note that this realization is slightly different from what we had in part (b) — instead of $\pm$ sign for the χ and the opposite $\mp$ sign for the $\tilde\chi$, we now have the same overall factor $\pm i$ for all the χ_i.

(e) Show that the free Lagrangian (13) is invariant under this symmetry if and only if the mass matrix M^{jk} is real.

If the mass matrix M^{jk} is complex, we can make it real via some unitary transform of fermions into each other, $\chi_i(x) \to U_i^{\;j} \chi_j(x)$. Consequently, the free Weyl fermions always have a CP symmetry, but its action on the original (un-transformed) spinors becomes

$$\mathbf{CP} : \chi_j(\mathbf{x}, t) \;\to\; \sum_k C_j^{\;k} \, \sigma_2 \chi_k^*(-\mathbf{x}, +t) \tag{16}$$

for some unitary matrix C.

(f) Show that the Lagrangian (13) is invariant under (16) provided the mass matrix M and the unitary matrix C are related by $CM^*C^\top = -M$.

⋆ For extra challenge, show that such C matrix exists for any complex symmetric mass matrix M.

However, for the interacting fermions, changing the basis and hence the CP action from (15) to (16) may spoil the CP symmetry of the interactions. In class, I shall explain how this happens for the weak interactions.