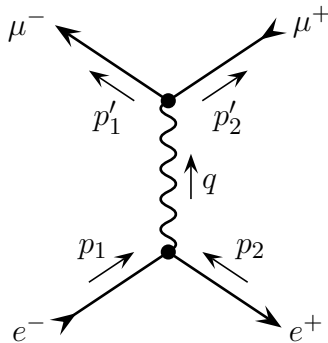


1. Consider the muon pair production in QED. At the tree level,



$$\langle \mu^-, \mu^+ | \mathcal{M} | e^-, e^+ \rangle = \frac{e^2}{s} \times \bar{u}(\mu^-) \gamma^\nu v(\mu^+) \times \bar{v}(e^+) \gamma_\nu u(e^-). \quad (1)$$

In class we have focused on the un-polarized cross-section for this process; in this exercise we focus on the polarized amplitudes for definite helicities of all 4 particles involved.

For simplicity, let us assume that all the particles are ultra-relativistic so that their Dirac spinors  $u(e^-)$ ,  $v(e^+)$ ,  $u(\mu^-)$ ,  $v(\mu^+)$  all have definite chiralities,

$$\begin{aligned} u_L &\approx \sqrt{2E} \begin{pmatrix} \xi_L \\ 0 \end{pmatrix}, & u_R &\approx \sqrt{2E} \begin{pmatrix} 0 \\ \xi_R \end{pmatrix}, \\ v_L &\approx -\sqrt{2E} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix}, & v_R &\approx \sqrt{2E} \begin{pmatrix} \eta_R \\ 0 \end{pmatrix}. \end{aligned} \quad (2)$$

cf. homework [set 8](#), eq. (8.16).

- (a) Show that in the approximation (2),

$$\bar{v}(e^+) \gamma_\nu u(e^-) = \bar{v}(e^+_R) \gamma_\nu u(e^-_L) = 0, \quad (3)$$

which means there is no muon pairs production unless the initial electron and positron have *opposite helicities*.

- (b) Show that the  $\mu^-$  and the  $\mu^+$  must also have *opposite helicities* because

$$\bar{u}(\mu^-) \gamma^\nu v(\mu^+) = \bar{u}(\mu^-_L) \gamma^\nu v(\mu^+_R) = 0. \quad (4)$$

- (c) Let's work in the center-of-mass frame where the initial  $e^-$  and  $e^+$  collide along the  $z$  axis,  $p_1^\nu = (E, 0, 0, +E)$ ,  $p_2^\nu = (E, 0, 0, -E)$ . Calculate the 4-vector  $\bar{v}(e^+) \gamma^\nu u(e^-)$  in

this frame and show that

$$\bar{v}(e_L^+) \gamma_\nu u(e_R^-) = 2E \times (0, +i, +1, 0), \quad \bar{v}(e_R^+) \gamma_\nu u(e_L^-) = 2E \times (0, -i, +1, 0). \quad (5)$$

- (d) In the CM frame the muons fly away in opposite directions at some angle  $\theta$  to the electron / positron directions. Without loss of generality we may assume the muons' momenta being in the  $xz$  plane, thus

$$p_1^\nu = (E, +E \sin \theta, 0, +E \cos \theta), \quad p_2^\nu = (E, -E \sin \theta, 0, -E \cos \theta) \quad (6)$$

Calculate the 4-vector  $\bar{u}(\mu^-) \gamma_\nu v(\mu^+)$  for the muons and show that

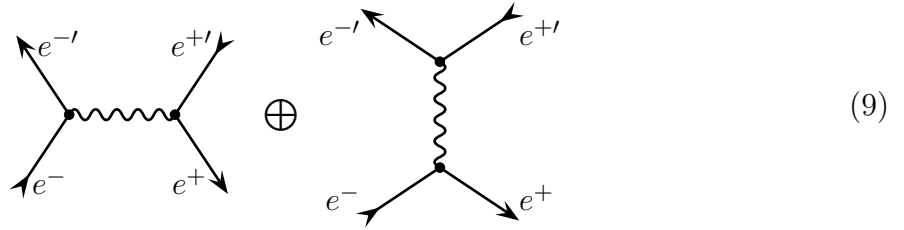
$$\begin{aligned} \bar{u}(\mu_R^-) \gamma^\nu v(\mu_L^+) &= 2E \times (0, -i \cos \theta, +1, +i \sin \theta), \\ \bar{u}(\mu_L^-) \gamma^\nu v(\mu_R^+) &= 2E \times (0, +i \cos \theta, +1, -i \sin \theta). \end{aligned} \quad (7)$$

- (e) Now calculate the amplitudes (1) for all possible combinations of particles' helicities, calculate the partial cross-sections, and show that

$$\begin{aligned} \frac{d\sigma(e_L^- + e_R^+ \rightarrow \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \rightarrow \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s} \times (1 + \cos \theta)^2, \\ \frac{d\sigma(e_L^- + e_R^+ \rightarrow \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \rightarrow \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s} \times (1 - \cos \theta)^2, \\ \frac{d\sigma(e_L^- + e_L^+ \rightarrow \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_R^+ \rightarrow \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} = 0, \\ \frac{d\sigma(e_{\text{any}}^- + e_{\text{any}}^+ \rightarrow \mu_L^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_{\text{any}}^- + e_{\text{any}}^+ \rightarrow \mu_R^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} = 0. \end{aligned} \quad (8)$$

- (f) Finally, sum / average over the helicities and calculate the un-polarized cross-section for the muon pair production.

2. Next, consider the *Bhabha scattering*  $e^-e^+ \rightarrow e^-e^+$ . In QED, there are two tree-level Feynman diagrams contributing to this process, namely



(a) Evaluate the two diagrams and write the amplitude  $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ . Mind the sign rules for the fermions.

Now comes the real work: calculating the un-polarized partial cross-section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} \quad (10)$$

where  $\overline{|\mathcal{M}|^2}$  stands for  $|\mathcal{M}|^2$  summed over final particle spins and averaged over the spins of the initial particles. Note the two diagrams (9) must be added together before squaring the amplitude, because

$$|\mathcal{M}_1 + \mathcal{M}_2|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2\text{Re}(\mathcal{M}_1^* \mathcal{M}_2) \neq |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2. \quad (11)$$

For simplicity, assume  $E \gg m_e$  and neglect the electron's mass throughout your calculation. You may find it convenient to express products of momenta in terms of Mandelstam's variables  $s$ ,  $t$ , and  $u$ . In the  $m_e \approx 0$  approximation,  $p_1^2 = p_2^2 = p_1'^2 = p_2'^2 = m_e^2 \approx 0$  while

$$(p_1 p_2) = (p_1' p_2') \approx \frac{1}{2}s, \quad (p_1 p_1') = (p_2 p_2') \approx -\frac{1}{2}t, \quad (p_1 p_2') = (p_2 p_1') \approx -\frac{1}{2}u. \quad (12)$$

(b) Let's start with the second diagram's amplitude  $\mathcal{M}_2$ . Sum / average the  $|\mathcal{M}_2|^2$  over all spins and show that

$$\frac{1}{4} \sum_{\text{all spins}} |\mathcal{M}_2|^2 = 2e^4 \times \frac{t^2 + u^2}{s^2}. \quad (13)$$

(c) Similarly, show that for the first diagram

$$\frac{1}{4} \sum_{\text{all spins}} |\mathcal{M}_1|^2 = 2e^4 \times \frac{s^2 + u^2}{t^2}. \quad (14)$$

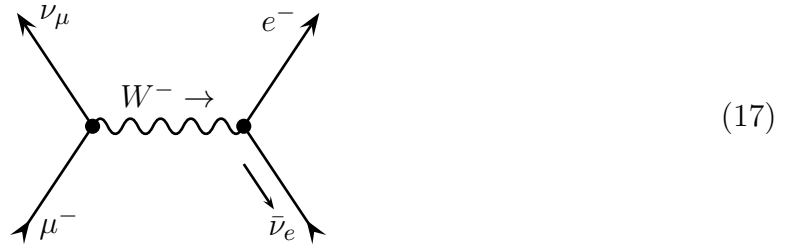
(d) Now consider the interference  $\mathcal{M}_1^* \times \mathcal{M}_2$  between the two diagrams. Show that

$$\frac{1}{4} \sum_{\text{all spins}} \mathcal{M}_1^* \times \mathcal{M}_2 = 2e^4 \times \frac{u^2}{st}. \quad (15)$$

(e) Finally assemble all the terms together and show that for the Bhabha scattering

$$\left( \frac{d\sigma}{d\Omega} \right)_{\text{c.m.}} = \frac{\alpha^2}{2s} \times \frac{s^4 + t^4 + u^4}{s^2 \times t^2} = \frac{\alpha^2}{4s} \times \left( \frac{3 + \cos^2 \theta}{1 - \cos \theta} \right)^2. \quad (16)$$

3. A muon usually decays into an electron, an electron-flavored antineutrino, and a muon-flavored neutrino,  $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$ . At the tree level of the Standard model, this decay proceeds through a single Feynman diagram



Since all the momenta in this diagram are much smaller than  $M_W$ , we may approximate the  $W$  propagator as simply  $ig^{\kappa\lambda}/M_W^2$ . Consequently, the decay amplitude is

$$\begin{aligned} \langle e^-, \bar{\nu}_e, \nu_\mu | \mathcal{M} | \mu^- \rangle &\approx \frac{ig^{\kappa\lambda}}{M_W^2} \times \bar{u}(\nu_\mu) \left( -ig_2 \gamma_\kappa \frac{1 - \gamma^5}{2} \right) u(\mu^-) \times \\ &\quad \times \bar{u}(e^-) \left( -ig_2 \gamma_\lambda \frac{1 - \gamma^5}{2} \right) v(\bar{\nu}_e) \\ &= \frac{G_F}{\sqrt{2}} \left[ \bar{u}(\nu_\mu) \gamma^\lambda (1 - \gamma^5) u(\mu^-) \right] \times \left[ \bar{u}(e^-) \gamma_\lambda (1 - \gamma^5) v(\bar{\nu}_e) \right] \end{aligned} \quad (18)$$

where  $G_F \approx 1.17 \cdot 10^{-5} \text{ GeV}^{-2}$  is the Fermi constant. In this exercise, you will use this amplitude to calculate the muon's net decay rate  $\Gamma$  and the energy spectrum  $d\Gamma/dE_e$  of the final state electrons.

- (a) Sum the absolute square of the amplitude (18) over the final particle spins and average over the initial muon's spin. Show that altogether,

$$\frac{1}{2} \sum_{\text{all spins}} |\langle e^-, \bar{\nu}_e, \nu_\mu | \mathcal{M} | \mu^- \rangle|^2 = 64G_F^2 (p_\mu \cdot p_{\bar{\nu}}) (p_e \cdot p_\nu). \quad (19)$$

The rest of this problem is the phase space calculation. The following lemma is very useful for three-body decays like  $\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e$ :

For a decay of initial particle of mass  $M_0$  into three final particles of respective masses  $m_1$ ,  $m_2$ , and  $m_3$ , the partial decay rate in the rest frame of the original particle is

$$d\Gamma = \frac{1}{2M_0} \times \overline{|\mathcal{M}|^2} \times \frac{d^3\Omega}{256\pi^5} \times dE_1 dE_2 dE_3 \delta(E_1 + E_2 + E_3 - M_0), \quad (20)$$

where  $d^3\Omega$  comprises three angular variables parametrizing the directions of the three final-state particles relative to some external frame, but not affecting the angles between the three momenta. For example, one may use two angles to describe the orientation of the decay plane (the three momenta are coplanar,  $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$ ) and one more angle to fix the direction of *e.g.*,  $\mathbf{p}_1$  in that plane. Altogether,  $\int d^3\Omega = 4\pi \times 2\pi = 8\pi^2$ .

- (b) Prove this lemma. Also show that when  $m_1 = m_2 = m_3 = 0$ , the kinematically allowed range of the final particles' energies is given by

$$0 \leq E_1, E_2, E_3 \leq \frac{1}{2}M_0 \quad \text{while} \quad E_1 + E_2 + E_3 = M_0, \quad (21)$$

but for the non-zero masses  $m_{1,2,3}$  this range is much more complicated.

Note that the electron and the neutrinos are much lighter than the muon, so in most decay events all three final-state particles are ultra-relativistic. This allows us to approximate  $m_e \approx m_\nu \approx m_{\bar{\nu}} \approx 0$ , which gives us the limits (21) for the final particles' energies.

Experimentally, the neutrinos and the antineutrinos are hard to detect. But it is easy to measure the muon's net decay rate  $\Gamma = 1/\tau_\mu$  and the energy distribution  $d\Gamma/dE_e$  of the electrons produced by decaying muons.

- (c) Integrate the muon's partial decay rate over the final particle energies and derive first the  $d\Gamma/dE_e$  and then the total decay rate.