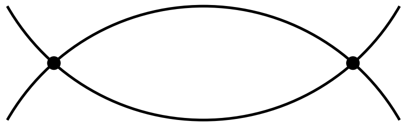
1. In class, we have evaluated the one-loop diagram



(1)

using the *hard-edge cutoff* as an ultraviolet regulator. Your task is to evaluate the same diagram using two other UV regulators: (1) Pauli–Villars, and (2) higher derivatives.

Show that all 3 regulators yield similar amplitudes of the form

$$\mathcal{M}(\text{diagram (1)}) \;=\; \frac{\lambda_{\text{bare}}^2}{32\pi^2}\times\left(\log\frac{\Lambda^2}{m^2}\;+\;C\;+\;J(t/m^2)\;+\;\text{negligible}\right) \tag{2}$$

where

$$J(t/m^2) \;=\; \int\limits_0^1 d\xi\,\log\frac{m^2}{m^2-t\xi(1-\xi)} \tag{3}$$

'negligible' stands for terms that vanish as negative powers of the cutoff scale Λ for $\Lambda\to\infty$, and C is an $O(1)$ numeric constant that depends on the particular UV regulator:

$$C_{\text{hard edge}}\;\neq\;C_{\text{Paili Villars}}\;\neq\;C_{\text{higher derivative}}\,. \tag{4}$$

Fortunately, this regulator dependence can be canceled by adjusting the cutoff scale parameter Λ for each regulator: Let

$$\Lambda_{\text{HE}}^2\times e^{C_{\text{HE}}}\;=\;\Lambda_{\text{PV}}^2\times e^{C_{\text{PV}}}\;=\;\Lambda_{\text{HE}}^2\times e^{C_{\text{HD}}}, \tag{5}$$

then all 3 regulators would yield exactly the same loop amplitude (2).

Note: the dimensional regularization also yields exactly the same amplitude (2), provided we identify the UV cutoff scale as

$$\Lambda_{\rm DR}^2 \;=\; \mu^2 \times \exp\left(\frac{1}{\epsilon} = \frac{2}{4-D}\right) \tag{6}$$

and then set

$$\Lambda_{\rm DR}^2 \times e^{C_{\rm DR}} \;=\; \Lambda_{\rm HE}^2 \times e^{C_{\rm HE}} \;=\; \Lambda_{\rm PV}^2 \times e^{C_{\rm PV}} \;=\; \Lambda_{\rm HE}^2 \times e^{C_{\rm HD}} \tag{7}$$

for a suitable $O(1)$ numeric constant $C_{\rm DR}$.

Hint: for the higher-derivative regulator, approximate the modified propagator as

$$\frac{i}{q^2 - m^2 - (q^4/\Lambda^2) + i\epsilon} \;\approx\; \frac{i}{q^2 - m_i^2\epsilon} \times \frac{-\Lambda^2}{q^2 - \Lambda^2 + i\epsilon} \tag{8}$$

where the second factor differs from 1 only for very large momenta. Consequently, for the two propagators in the loop we may further approximate

$$\frac{-\Lambda^2}{q_1^2 - \Lambda^2 + i\epsilon} \;\approx\; \frac{-\Lambda^2}{q_2^2 - \Lambda^2 + i\epsilon} \;\approx\; \frac{-\Lambda^2}{(q_1 - \xi q_{\rm net})^2 - \Lambda^2 + i\epsilon}\,. \tag{9}$$

2. Verify the integrals used by the Feynman's parameter trick and its generalizations:

$$\frac{1}{AB} \;=\; \int_0^1 \frac{d\xi}{[\xi A + (1-\xi)B]^2}\,, \tag{F.a}$$

$$\frac{1}{A^n B} \;=\; \int_0^1 \frac{n\xi^{n-1}d\xi}{[\xi A + (1-\xi)B]^{n+1}}\,, \tag{F.b}$$

$$\frac{1}{A^n B^m} \;=\; \frac{(n+m-1)!}{(n-1)!(m-1)!} \times \int_0^1 \frac{\xi^{n-1}(1-\xi)^{m-1}d\xi}{[\xi A + (1-\xi)B]^{n+m}}\,, \tag{F.c}$$

$$
\begin{aligned}
\frac{1}{ABC} &= \int\limits_0^1 d\xi \int\limits_0^{1-\xi} \frac{2d\eta}{[\xi A + \eta B + (1-\xi-\eta)C]^3} \\
&\equiv \iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi\, d\eta\, d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{2}{[\xi A + \eta B + (1-\xi-\eta)C]^3}\,, \qquad \text{(F.d)} \\
\frac{1}{A_1 A_2 \cdots A_k} &= \int\limits_{\xi_1,\ldots,\xi_k\geq 0} \!\!\cdots\!\! \int d^k\xi\, \delta(\xi_1+\cdots+\xi_k-1) \times \frac{(k-1)!}{[\xi_1 A_1 + \cdots + \xi_k A_k]^k}\,, \qquad \text{(F.e)} \\
\frac{1}{A_1^{n_1} A_2^{n_2} \cdots A_k^{n_k}} &= \frac{(n_1+\cdots+n_k-1)!}{(n_1-1)!\cdots(n_k-1)!} \times \qquad \text{(F.f)} \\
&\times \int\limits_{\xi_1,\ldots,\xi_k\geq 0} \!\!\cdots\!\! \int d^k\xi\, \delta(\xi_1+\cdots+\xi_k-1) \times \frac{\xi_1^{n_1-1}\cdots\xi_k^{n_k-1}}{[\xi_1 A_1 + \cdots + \xi_k A_k]^{(n_1+\cdots+n_k)}}\,.
\end{aligned}
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