1. First, a reading assignment: §7.2 of the Peskin & Schroeder textbook about the *LSZ reduction formula.*

2. Next, a simple exercise about the Yukawa theory. For $M_s > 2m_f$, the scalar particle becomes unstable: it decays into a fermion and an antifermion, $S \to f + \bar{f}$.

   (a) Calculate the tree-level decay rate $\Gamma(S \to f + \bar{f})$.

   (b) In class, we have calculated

$$\Sigma_\Phi^{1\,\mathrm{loop}}(p^2) = \frac{12g^2}{16\pi^2}\int_0^1 d\xi\,\Delta(\xi) \times \left[\frac{1}{\epsilon} - \gamma_E + \frac{1}{3} + \log\frac{4\pi\mu^2}{\Delta(\xi)}\right] \tag{1}$$

$$\text{for } \Delta(\xi) = m_f^2 - \xi(1-\xi)p^2. \tag{2}$$

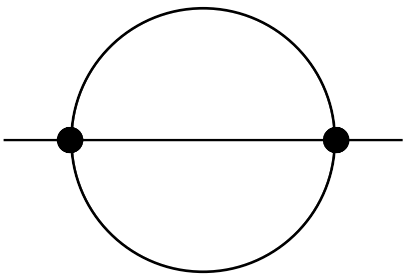
   Show that for $p^2 > 4m_f^2$, this $\Sigma_\Phi(p^2)$ has an imaginary part and calculate it for $p^2 = M_s^2 + i\epsilon$.
   Note: at this level, you may neglect the difference between m_f^{bare} and m_f^{physical}.

   (c) Verify that

$$\mathrm{Im}\,\Sigma_\Phi^{1\,\mathrm{loop}}(p^2 = M_s^2 + i\epsilon) = -M_s\Gamma^{\mathrm{tree}}(S \to f + \bar{f}) \tag{3}$$

   and explain this relation in terms of the optical theorem.

3. Finally, a harder exercise about the scalar $\lambda\phi^4$ theory. As discussed in class, in this theory field strength renormalization begins at two-loop level. Specifically, the 1PI diagram



(4)

   provides the leading contribution to the $d\Sigma(p^2)/dp^2$ and hence to the $Z - 1$. *Your task is to evaluate this contribution.*

(a) First, write the two-loop $\Sigma(p^2)$ as an integral over two independent loop momenta, say q_1^μ and q_2^μ, then use Feynman's parameter trick — *cf.* eq. (F.d) of the homework set 13 — to write the product of three propagators as

$$\iiint d\xi\, d\eta\, d\zeta\, \delta(\xi+\eta+\zeta-1)\, \frac{2}{(\mathcal{D})^3} \tag{5}$$

where $\mathcal{D}$ is a quadratic polynomial of the momenta q_1, q_2, p. Finally, change the independent momentum variables from q_1 and q_2 to $k_1 = q_1 + \text{something} \times q_2 + \text{something} \times p$ and $k_2 = q_2 + \text{something} \times p$ to give $\mathcal{D}$ a simpler form

$$\mathcal{D} \;=\; \alpha \times k_1^2 \;+\; \beta \times k_2^2 \;+\; \gamma \times p^2 \;-\; m^2 \;+\; i0 \tag{6}$$

for some (ξ,η,ζ)–dependent coefficients α, β, γ, for example

$$\alpha \;=\; (\xi+\zeta), \qquad \beta \;=\; \frac{\xi\eta+\xi\zeta+\eta\zeta}{\xi+\zeta}, \qquad \gamma \;=\; \frac{\xi\eta\zeta}{\xi\eta+\xi\zeta+\eta\zeta}. \tag{7}$$

Make sure the momentum shift has unit Jacobian $\partial(q_1,q_2)/\partial(k_1,k_2) = 1$.

Warning: Do not set $p^2 = m^2$ at this stage.

(b) Express the derivative $d\Sigma(p^2)/dp^2$ in terms of

$$\iint d^4k_1\, d^4k_2\, \frac{1}{\mathcal{D}^4}. \tag{8}$$

Note that although this momentum integral diverges as $k_{1,2} \to \infty$, the divergence is logarithmic rather than quadratic.

(c) To evaluate the momentum integral (8), Wick-rotate the momenta k_1 and k_2 to the Euclidean space, and then use the dimensional regularization. Here are some useful formulæ for this calculation:

$$\frac{6}{A^4} \;=\; \int_0^\infty dt\, t^3\, e^{-At}, \tag{9}$$

$$\int \frac{d^Dk}{(2\pi)^D}\, e^{-ctk^2} \;=\; \bigl(4\pi ct\bigr)^{-D/2}, \tag{10}$$

$$\Gamma(2\epsilon)X^\epsilon \;=\; \frac{1}{2\epsilon} \;-\; \gamma_E \;+\; \tfrac{1}{2}\log X \;+\; O(\epsilon). \tag{11}$$

(d) Assemble your results as

$$\frac{d\Sigma(p^2)}{dp^2} = -\frac{\lambda^2}{12(4\pi)^4}\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\,\delta(\xi+\eta+\zeta-1)\times\frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3}\times$$
$$\times\left(\frac{1}{\epsilon} - 2\gamma_E + 2\log\frac{4\pi\mu^2}{m^2} + \log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta+\xi\zeta+\eta\zeta-\xi\eta\zeta(p^2/m^2))^2}\right). \tag{12}$$

(e) Before you evaluate the Feynman parameter integral (12) — which looks like a frightful mess — make sure it does not introduce its own divergences. That is, without actually calculating the integrals

$$\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\,\delta(\xi+\eta+\zeta-1)\times\frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3}, \tag{13}$$
$$\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\,\delta(\xi+\eta+\zeta-1)\times\frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3}\times\log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta+\xi\zeta+\eta\zeta-\xi\eta\zeta(p^2/m^2))^2},$$

make sure that they converge. Pay attentions to the boundaries of the parameter space and especially to the corners where $\xi,\eta\to 0$ while $\zeta\to 1$ (or $\xi,\zeta\to 0$, or $\eta,\zeta\to 0$).

This calculation shows that

$$\frac{d\Sigma}{dp^2} = \frac{\text{constant}}{\epsilon} + \textit{a\_finite\_function}(p^2) \tag{14}$$

and hence

$$\Sigma(p^2) = (\text{a divergent constant}) + (\text{another divergent constant})\times p^2 + \textit{a\_finite\_function}(p^2) \tag{15}$$

up to the two-loop order. In fact, this behavior persists to all loops, so all the divergences of $\Sigma(p^2)$ may be canceled with just two counterterms, δ^m and $\delta^Z\times p^2$.

⋆ Optional exercise: Evaluate the integrals (13) for $p^2 = m^2$ and show that

$$\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\,\delta(\xi+\eta+\zeta-1)\times\frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3} = \frac{1}{2}, \tag{16}$$
$$\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\,\delta(\xi+\eta+\zeta-1)\times\frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3}\times\log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta+\xi\zeta+\eta\zeta-\xi\eta\zeta)^2} = -\frac{3}{4}.$$

Do not try to do this calculation by hand — it would take way too much time. Instead, use *Mathematica* or equivalent software. To help it along, replace the (ξ,η,ζ) variables with (x,w) according to

$$\begin{gathered} \xi = w\times x, \quad \eta = w\times(1-x), \quad \zeta = 1-w, \\ \iiint d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) = \int_0^1 dx \int_0^1 dw\, w, \end{gathered} \tag{17}$$

then integrate over w first and over x second.

Alternatively, you may evaluate the integrals like this numerically. In this case, don't bother changing variables, just use a simple 2D grid spanning a triangle defined by $\xi+\eta+\zeta=1$, $\xi,\eta,\zeta\geq 0$; modern computers can sum up a billion grid points in less than a minute. But watch out for singularities at the corners of the triangle.

(f) Finally, calculate the field strength renormalization factor

$$Z = \left[1 - \frac{d\Sigma}{dp^2}\right]^{-1} \tag{18}$$

to the two-loop order. Use the bare perturbation theory, *i.e.* divergent λ_{bare} and m^2_{bare} instead of the counterterms.

Note: the derivative $d\Sigma/dp^2$ in eq. (18) should be evaluated at $p^2 = M^2_{\text{ph}}$ — the physical mass^2 of the scalar particle, but to the leading approximation we may let $M^2_{\text{ph}} \approx m^2$ and set $p^2 = m^2$ in eq. (12). This simplifies the second integral (13) a little bit — *cf.* eqs. (16) — although it's still a royal pain to calculate.