1. The first problem is about the muon's gyromagnetic factor g. Experimentally, it has been measured with a very high precision (9 significant digits) and the theoretical calculations have a similarly high precision. At present, there is a very small discrepancy

$$g_\mu^{\rm exp} \;-\; g_\mu^{\rm theory} \;\approx\; (58 \pm 13^{\rm stat} \pm 12^{\rm syst}) \cdot 10^{-10}, \tag{1}$$

which is probably due new physics beyond the Minimal Standard Model (*i.e.*, loops involving some new particles, for example the superpartners). However, there is also some uncertainty in the photon-hadron coupling, which affects the g_μ at the two-loop order in QED. Usually, the photon-hadron coupling is obtained from the $e^+e^\to$hadrons cross-section, but it can also be derived from the vector spectral functions obtained from the $\tau \to \nu_\tau +$ hadrons decays, and the two method yield slightly different results: the former leads to eq. (1) while the later leads to a smaller discrepancy

$$g_\mu^{\rm exp} \;-\; g_\mu^{\rm theory} \;\approx\; (15 \pm 13^{\rm stat} \pm 12^{\rm syst}) \cdot 10^{-10}. \tag{2}$$

In this exercise we consider non-minimal versions of the Standard Model which contain some extra particles. Your task is to calculate the effect of such particles on the muon's magnetic moment at the one-loop level, and comparing your results to eq. (1) establish limits on masses and couplings of those extra particles.

(a) Let's extend the minimal standard model by adding just one extra field Φ, a heavy neutral scalar of mass $M_S \simeq 200$ GeV which has a Yukawa coupling to the muon field Ψ,

$$\mathcal{L} \;\supset\; g\Phi \times \overline{\Psi}\Psi. \tag{3}$$

Calculate Φ's contribution to the muon's magnetic moment at the one-loop level. Then use your result to derive an upper limit on the Yukawa coupling g.

(b) A different extension of the Standard model contains an *axion*, a very light pseudoscalar field ϕ which couples to muons (and other leptons) according to

$$\mathcal{L} \supset \frac{\partial_\mu \phi}{f_a} \times \overline{\Psi}\gamma^5\gamma^\mu\Psi \approx \frac{2im_{\text{muon}}}{f_a}\,\phi \times \overline{\Psi}\gamma^5\Psi + \text{ a total derivative.} \tag{4}$$

The axion is a pseudo-Goldstone boson resulting from spontaneous breakdown of an axial symmetry at a very high energy scale $f_a \gg 100$ GeV; the symmetry is inexact but very good, so the axion's mass is non-zero but very small, $M_A \ll 1$ MeV.
Calculate the axion's contribution to the muon's magnetic moment at the one-loop level. Then use your result to derive a lower limit on the axion scale f_a.

2. Next, calculate the δ_2 counterterm of QED at the one-loop level and verify that it equals to the δ_1 counterterm we have calculated in class — *cf.* eq. (80) of my notes — including the finite parts of both counterterms.

   The counterterms depend on the regulators (both UV and IR) and on the gauge used for the photon propagators, so use the same gauge and regulators we have used in class: $D = 4 - 2\epsilon < 4$ dimensions to regulate the UV divergence, a tiny photon mass $m_\gamma^2 \ll m_e^2$ to regulate the IR divergence, and the Feynman gauge for the photon propagators. Start by calculating the $\Sigma^{1,\text{loop}}(\not{p})$ for the off-shell electron momenta p, then take the derivative $d\Sigma/d\not{p}$, and only then take the momentum on-shell, $\not{p} \to m_e$. Note that $\Sigma(\not{p})$ itself is infrared-finite, but its derivative has an IR singularity when the momentum goes on-shell, and that's why you need the IR regulator.

   Note: You should get $\delta_2 = \delta_1$ *before you take the $D \to 4$ limit.* If this does not work, check your calculations for mistakes.

3. Finally, consider the "scalar QED" — the theory of EM fields A^μ coupled to a charged scalar field Φ instead of the electron field Ψ. The counterterms of this theory are related by two Ward identities:

$$\delta_2 = \delta_1^{(1\gamma)} = \delta_1^{(2\gamma)}. \tag{5}$$

   Your task is to prove these identities using the Ward–Takahashi identities you should have proved in the previous homework.

Note: the renormalization condition for the finite part of the δ_2 counterterm is similar to the δ_Z in the $\lambda\phi^4$ theory, but the conditions for the $\delta_1^{(1\gamma)}$ and $\delta_1^{(2\gamma)}$ counterterms are more complicated. Similar to the ordinary QED, they follow from the net electric charge of the scalar particle being exactly q, without any quantum corrections. But for the scalar particle, we need two physical processes to measure the net charge, namely the low-momentum Coulomb scattering, and also the Thompson scattering of low-frequency photons.

Let's rephrase these conditions in terms of the 1PI amplitudes for two scalars and one or two photons. Let $G^\mu(p',p)$ be the net 1PI amplitude for two scalars and one photon, including the tree-level vertex, the loop corrections, and the $\delta_1^{(1\gamma)}$ counterterm. Likewise, let $G^{\mu\nu}(p',p;k_1,k_2)$ be the net 1PI amplitude for two scalars and two photons, also including the tree-level vertex, the loop corrections, and the $\delta_1^{(2\gamma)}$ counterterm. Take the limit of zero photon momenta, then by Lorentz symmetry

$$\begin{aligned} G^\mu(p'=p\,;k=0) &= -iq(p'+p)^\mu \times A(p^2), \\ G^{\mu\nu}(p'=p\,;k_1=k_2=0) &= iq^2 g^{\mu\nu}\times B(p^2) \;+\; iq^2 p^\mu p^\nu \times C(p^2). \end{aligned} \tag{6}$$

The net electric charge measured by the $k\to 0$ limit of the Coulomb scattering is $q\times A(p^2=m^2)$, while the charge^2 measured by the Thompson scattering is $q^2\times\frac{1}{2}B(p^2=m^2)$. Thus, to avoid quantum corrections to the physical electric charge we need

$$\text{for } p^2 = m^2, \quad A = 1 \quad \text{and} \quad B = 2. \tag{7}$$

The finite parts of the $\delta_1^{(1\gamma)}$ and $\delta_1^{(2\gamma)}$ counterterms follow from these on-shell conditions.