

1. First, a bit of group theory. Consider a generic simple non-abelian compact Lie group G and its generators T^a . For a suitable normalization of the generators,

$$\mathrm{tr}_{(r)}(T^a T^b) \equiv \mathrm{tr} \left(T_{(r)}^a T_{(r)}^b \right) = R(r) \delta^{ab} \quad (1)$$

where the trace is taken over any complete multiplet (r) — irreducible or reducible, it does not matter — and $T_{(r)}^a$ is the matrix representing the generator T^a in that multiplet. The coefficient $R(r)$ in eq. (1) depends on the multiplet (r) but it's the same for all generators T^a and T^b . The $R(r)$ is called the *index* of the multiplet (r) .

The (quadratic) Casimir operator $C_2 = \sum_a T^a T^a$ commutes with all the generators, $\forall b, [C_2, T^b] = 0$. Consequently, when we restrict this operator to any *irreducible* multiplet (r) of the group G it becomes a unit matrix times some number $C(r)$. In other words,

$$\text{for an irreducible } (r), \quad \sum_a T_{(r)}^a T_{(r)}^a = C(r) \times \mathbf{1}_{(r)}. \quad (2)$$

For example, for the isospin group $SU(2)$, the Casimir operator is $C_2 = \vec{I}^2$, the irreducible multiplets have definite isospin $I = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$, and $C(I) = I(I + 1)$.

- (a) Show that for any irreducible multiplet (r) ,

$$\frac{R(r)}{C(r)} = \frac{\dim(r)}{\dim(G)}. \quad (3)$$

In particular, for the $SU(2)$ group, this formula gives $R(I) = \frac{1}{3}I(I + 1)(2I + 1)$.

- (b) Suppose the first three generators of G generate an $SU(2)$ subgroup. Show that if a multiplet (r) of G decomposes into several $SU(2)$ multiplets of isospins I_1, I_2, \dots, I_n , then

$$R(r) = \sum_{i=1}^n \frac{1}{3} I_i (I_i + 1) (2I_i + 1). \quad (4)$$

- (c) Now consider the $SU(N)$ group with an obvious $SU(2)$ subgroup of matrices acting only on the first two components of a complex N -vector. This complex N -vector is

called the fundamental multiplet (of the $SU(N)$) and denoted (N) or \mathbf{N} . As far as the $SU(2)$ subgroup is concerned, (N) comprises one doublet and $N - 2$ singlets, hence

$$R(N) = \frac{1}{2} \quad \text{and} \quad C(N) = \frac{N^2 - 1}{2N}. \quad (5)$$

Show that the adjoint multiplet of the $SU(N)$ decomposes into one $SU(2)$ triplet, $2(N - 2)$ doublets, and $(N - 2)^2$ singlets, therefore

$$R(\text{adj}) = C(\text{adj}) \equiv C(G) = N. \quad (6)$$

Hint: $(N) \times (\bar{N}) = (\text{adj}) + (1)$.

- (d) The symmetric and the anti-symmetric 2-index tensors form irreducible multiplets of the $SU(N)$ group. Find out the decomposition of these multiplets under the $SU(2) \subset SU(N)$ and calculate their respective indices R and Casimirs C .

2. Now let's apply this group theory to physics. Consider quark-antiquark pair production in QCD, specifically $u\bar{u} \rightarrow d\bar{d}$. There is only one tree diagram contributing to this process,



Evaluate this diagram, then sum/average the $|\mathcal{M}|^2$ over both spins and *colors* of the final/initial particles to calculate the total cross section. For simplicity, you may neglect the quark masses.

Note that the diagram (7) looks exactly like the QED pair production process $e^-e^+ \rightarrow \text{virtual } \gamma \rightarrow \mu^-\mu^+$, so you can re-use the QED formula for summing/averaging over the spins. But in QCD, you should also sum/average over colors of all the quarks, and that's the whole point of this exercise.

3. In problem 2 from [previous homework set](#) you should have calculated the annihilation of a scalar ‘quark’ and an ‘antiquark’ into a pair of gluons. To convert the annihilation amplitude into a cross-section we need to sum/average over the colors of all the particles. As a first step in this direction, it’s convenient to write the amplitude as

$$\mathcal{M}(j + i \rightarrow a + b) = F \times \{T^a, T^b\}_j^i + iG \times [T^a, T^b]_j^i \quad (8)$$

where F and G are some functions of momenta and polarizations of the vector particles while $a, b, i,$ and j are the color indices of the four particles. Specifically, the a and b colors of the gauge bosons run over the adjoint multiplet of G , the j index of the scalar ‘quark’ runs over the multiplet (r), and the i index of the scalar ‘antiquark’ runs over the conjugate multiplet (\bar{r}).

- (a) Show that the annihilation amplitude indeed has form (8) and write down the coefficients F and G as explicit functions of the particles momenta and polarizations.
- (b) Next, let us sum the $|\mathcal{M}|^2$ over the gauge boson’s colors and average over the scalars’ colors. Show that

$$\frac{1}{\dim^2(r)} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 = \frac{C(r)}{\dim(r)} \times (4C(r) \times |F|^2 + C(\text{adj}) \times (|G|^2 - |F|^2)). \quad (9)$$

In particular, for scalars in the fundamental representation of the $SU(N)$ gauge group,

$$\frac{1}{N^2} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 = \frac{N^2 - 1}{2N^2} \left(\frac{N^2 - 2}{N} |F|^2 + N|G|^2 \right). \quad (10)$$

- (c) Evaluate F and G in the center of mass frame, where the vector particles’ polarizations $e_{1,2}^\mu = (0, \mathbf{e}_{1,2})$ are purely spatial and transverse to the vectors’ momenta $\pm \mathbf{k}$. For simplicity, use planar rather than circular polarizations.
- (d) Assemble your results and calculate the (polarized, partial) cross-section for the annihilation process.

4. In class, I calculated the (infinite parts of the) δ_2 and δ_1 counterterms for the quarks. Your task is to calculate the analogous $\delta_2^{(\text{gh})}$ and $\delta_1^{(\text{gh})}$ counterterms for the *ghosts fields*.
- (a) Draw one-loop diagrams whose divergences are cancelled by the $\delta_2^{(\text{gh})}$ and $\delta_1^{(\text{gh})}$, and calculate the group factors for each diagrams.
 - (b) Calculate the momentum integrals for the diagrams. Focus on the UV divergences and ignore the finite parts of the integrals.
 - (c) Assemble your results and show that the *difference* $\delta_1^{(\text{gh})} - \delta_2^{(\text{gh})}$ for the ghosts is exactly the same as the $\delta_1 - \delta_2$ difference for the quarks.