1. The first exercise is about the b_1, b_2, b_3 coefficients of the one-loop beta-functions

$$\beta_1^{\text{1 loop}} = \frac{b_1 g_1^3}{16\pi^2}, \quad \beta_2^{\text{1 loop}} = \frac{b_2 g_2^3}{16\pi^2}, \quad \beta_3^{\text{1 loop}} = \frac{b_3 g_3^3}{16\pi^2} \tag{1}$$

for the three gauge couplings of the $SU(3) \times SU(2) \times U(1)$ Standard Model.

(a) Calculate the b_1, b_2, b_3 it for the minimal version of the Standard Model: the $SU(3) \times SU(2) \times U(1)$ gauge fields, one Higgs doublet, three families of quarks and leptons, and nothing else.

⋆ FYI, each family comprises 8 left-handed Weyl fields in the $(\mathbf{3}, \mathbf{2}, y = +\frac{1}{6})$ and $(\mathbf{1}, \mathbf{2}, y = -\frac{1}{2})$ multiplets of the gauge symmetry and 7 right-handed Weyl fermions in the $(\mathbf{3}, \mathbf{1}, y = +\frac{2}{3})$, $(\mathbf{3}, \mathbf{1}, y = -\frac{1}{3})$, and $(\mathbf{1}, \mathbf{1}, y = -1)$ multiplets.

(b) Re-calculate the b_1, b_2, b_3 for the MSSM — the Minimal Supersymmetric Standard Model. FYI, here is complete list of the MSSM fields:

◦ The $SU(3) \times SU(2) \times U(1)$ gauge fields, same as the non-SUSY SM.

• For each vector field there is a Majorana fermion (gaugino) with similar $SU(3) \times SU(2) \times U(1)$ quantum numbers. Altogether, there is an adjoint multiplet of gauginos for each factor of the gauge symmetry.

◦ 3 families of quarks and leptons, same as the non-SUSY SM.

• For each Weyl fermion — left-handed or right-handed — in these three families, the MSSM also have a complex scalar field (squark or slepton) with similar $SU(3) \times SU(2) \times U(1)$ quantum numbers. Altogether, this makes 45 complex scalar fields in similar multiplets to the quarks and leptons.

• The Higgs sector of the MSSM comprises *two* $SU(2)$ doublets of complex scalars accompanied by one $SU(2)$ doublet of Dirac fermions (the higgsinos); all these doublets have $y = \frac{1}{2}$.

— There are all kinds of Yukawa and ϕ^4 interactions between the MSSM fields, but you do not need them for the one-loop calculation of the gauge couplings' beta-functions.

In Grand Unified Theories

$$\alpha_3 \;=\; \alpha_2 \;=\; \tfrac{5}{3}\alpha_1 \;=\; \alpha_{\rm GUT} \quad \text{at the GUT scale.} \tag{2}$$

At lower energy scales $E \ll M_{\rm GUT}$ the SM couplings are given (lo the leading one-loop order) by the Georgi–Quinn–Weinberg equations

$$\begin{aligned}
\frac{1}{\alpha_3(E)} &= \frac{1}{\alpha_{\rm GUT}} \;+\; b_3 \times \frac{1}{2\pi}\log\frac{M_{\rm GUT}}{E}\,,\\
\frac{1}{\alpha_2(E)} &= \frac{1}{\alpha_{\rm GUT}} \;+\; b_2 \times \frac{1}{2\pi}\log\frac{M_{\rm GUT}}{E}\,,\\
\frac{1}{\alpha_1(E)} &= \frac{5/3}{\alpha_{\rm GUT}} \;+\; b_1 \times \frac{1}{2\pi}\log\frac{M_{\rm GUT}}{E}\,.
\end{aligned} \tag{3}$$

(c) Derive these equations from eqs. (1).

(d) Experimentally, at $E = M_Z \approx 91$ GeV

$$\frac{1}{\alpha_3(M_Z)} \approx 8.5, \quad \frac{1}{\alpha_2(M_Z)} \approx 29.6, \quad \frac{1}{\alpha_1(M_Z)} \approx 98.4. \tag{4}$$

Check that these couplings are consistent with eqs. (3) for the MSSM but not for the non-SUSY minimal Standard Model. For the MSSM, calculate the Grand Unification scale $M_{\rm GUT}$ and the unified gauge coupling $\alpha_{\rm GUT}$.
For simplicity, use the b_3, b_2, b_1 coefficients of the massless theory — the minimal SM or the MSSM — for all energies between M_Z and $M_{\rm GUT}$.

2. Next, a reading assignment: $16.7 of *Peskin & Schroeder* about the "magnetic anti-screening" explanation of the asymptotic freedom in QCD.

3. Another reading assignment: §19.3 of *Peskin & Schroeder* about the chiral symmetry of QCD and the pions.

   Chapter 19 of *Weinberg* has a deeper discussion of pions (and Goldstone bosons in general); you are advised to read it, but not necessarily this week.

4. The f_π is called the *pion decay constant* because it controls the decay rate of the charged pions, mostly into muons and neutrinos, $\pi^+ \to \mu^+\nu_\mu$ and $\pi^- \to \mu^-\bar{\nu}_\mu$.

The weak interactions at energies $O(M_\pi) \ll M_W$ are governed by the Fermi's current-current effective Lagrangian

$$\mathcal{L} \;=\; -2\sqrt{2}G_F J_L^{+\alpha} J_{L\alpha}^{-} \tag{5}$$

where $L_L^{\pm\alpha} = \frac{1}{2}(J_V^{\pm\alpha} - J_A^{\pm\alpha})$ are the left-handed charged currents. In terms of the quark and lepton fields,

$$\begin{aligned} J_L^{+\alpha} &= \tfrac{1}{2}\overline{\Psi}(\nu_\mu)(1-\gamma^5)\gamma^\alpha\Psi(\mu) \;+\; \cos\theta_c \times \tfrac{1}{2}\overline{\Psi}(u)(1-\gamma^5)\gamma^\alpha\Psi(d) \;+\; \cdots, \\ J_L^{-\alpha} &= \tfrac{1}{2}\overline{\Psi}(\mu)(1-\gamma^5)\gamma^\alpha\Psi(\nu_\mu) \;+\; \cos\theta_c \times \tfrac{1}{2}\overline{\Psi}(d)(1-\gamma^5)\gamma^\alpha\Psi(u) \;+\; \cdots, \end{aligned} \tag{6}$$

where the $\cdots$ stand for other fermions of the Standard Model, and $\theta_c \approx 13^\circ$ is the Cabibbo angle.

For the pion decay process, the axial part one of the currents annihilates the charged pion

$$\langle\text{vacuum}|\,\overline{\Psi}_d\gamma^5\gamma^\alpha\Psi_u\,\big|\pi^+(p)\big\rangle \;=\; \langle\text{vacuum}|\,\overline{\Psi}_u\gamma^5\gamma^\alpha\Psi_d\,\big|\pi^-(p)\big\rangle \;=\; -ip^\alpha f_\pi \tag{7}$$

while the other current creates the lepton pair.

(a) Show that the tree-level pion decay amplitude is

$$\mathcal{M}(\pi^+ \to \mu^+\nu_\mu) \;=\; \frac{G_f f_\pi \cos\theta_c}{\sqrt{2}} \times p^\alpha(\pi) \times \bar{u}(\nu_\mu)(1-\gamma^5)\gamma_\alpha v(\mu^+). \tag{8}$$

(b) Sum over the fermion spins and calculate the decay rate $\Gamma(\pi^+ \to \mu^+\nu_\mu)$. FYI, $f_\pi \approx 130$ MeV, $M_\pi \approx 140$ MeV, $M_\mu \approx 106$ MeV, and $G_F \approx 1.17 \cdot 10^{-5}\,\text{GeV}^{-2}$.

(c) The charged pions decay to muons much more often than they decay to electrons,

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} \;=\; \frac{M_e^2}{M_\mu^2}\,\frac{(1-(M_e/M_\pi)^2)^2}{(1-(M_\mu/M_\pi)^2)^2} \;\approx 1.2 \cdot 10^{-4}. \tag{9}$$

Derive this formula, then explain this preference for the heavier final-state lepton in terms of mis-match between lepton's chirality and helicity.