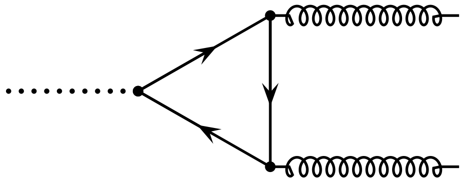
1. Consider the axial anomaly in a non-abelian gauge theory, for example QCD with N_f massless quark flavors,

$$J^\mu_A \;=\; \sum_{i,f} \overline{\Psi}_{if}\gamma^5\gamma^\mu\Psi^{if}, \qquad \partial_\mu J^\mu_A \;=\; -\frac{N_f g^2}{16\pi^2}\,\epsilon^{\alpha\beta\mu\nu}\,\mathrm{tr}\Big(F_{\alpha\beta}F_{\mu\nu}\Big) \tag{1}$$

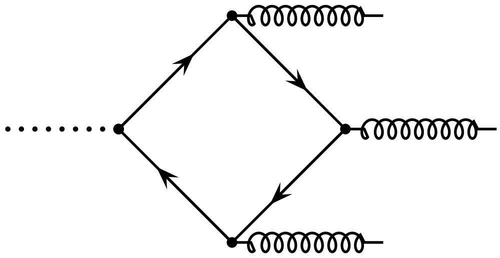
where $F_{\mu\nu}$ is the non-abelian gauge field strength.

(a) Expand the right hand side of eq. (1) into 2–gluon, 3–gluon, and 4–gluon terms and show that the 4–gluon term vanishes identically.
Hint: Use the cyclic symmetry of the trace.

The two-gluon anomaly term obtains from the triangle diagrams

 + gluon permutation. (2)

This works exactly as discussed in class for the QED, except in QCD we should trace $F_{\alpha\beta}F_{\gamma\delta}$ over the quark colors. But in QCD there is also the three-gluon anomaly (*cf.* part (a)) which obtains from the quadrangle diagrams

 + gluon permutations. (3)

(b) Evaluate the quadrangle diagrams using the Pauli–Villars regularization and derive the three-gluon anomaly in QCD. Note that for the regulators

$$\frac{1}{\not p - M}\,\gamma^5\,\not q\,\frac{1}{\not p + \not q - M} \;=\; \gamma^5\left(\frac{1}{\not p + \not q - M} - \frac{1}{\not p - M}\right) - 2M\gamma^5\,\frac{1}{\not p - M}\,\frac{1}{\not p + \not q - M}\,. \tag{4}$$

2. Next, a reading assignment: §22.2–3 of *Weinberg* about the chiral anomaly. Pay particular attention to the Jacobian of the fermion path integral and to regularization of the functional trace.

3. In any *even* spacetime dimension $d = 2n$, a massless Dirac fermion has an axial symmetry $\Psi(x) \to \exp(i\theta\Gamma)\Psi(x)$ where Γ generalizes the γ^5. Classically, the axial current $J_A^\mu = \overline{\Psi}\Gamma\gamma^\mu\Psi$ is conserved, but when the fermion is coupled to a gauge field — abelian or non-abelian — the axial symmetry is broken by the anomaly and

$$\partial_\mu J_A^\mu \;=\; -\frac{2}{n!}\left(\frac{g}{4\pi}\right)^n \epsilon^{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}\,\mathrm{tr}\Big(F_{\alpha_1\beta_1}F_{\alpha_2\beta_2}\cdots F_{\alpha_n\beta_n}\Big). \tag{5}$$

Generalize Weinberg's calculation of the anomaly via Jacobian of the fermionic path integral to any even spacetime dimension $d = 2n$.

For your information, in $2n$ Euclidean dimensions $\{\gamma^\mu,\gamma^\nu\} = +2\delta^{\mu\nu}$, $\Gamma = i^{n-2}\gamma^1\gamma^2\cdots\gamma^{2n}$, $\{\Gamma,\gamma^\mu\} = 0$, $\Gamma^2 = +1$, and $\mathrm{tr}(\Gamma\gamma^\alpha\gamma^\beta\cdots\gamma^\omega) = 2^n i^{2-n}\epsilon^{\alpha\beta\cdots\omega}$ (for $2n = d$ matrices $\gamma^\alpha\cdots\gamma^\omega$).

4. In any even dimension $d = 2n$, the right hand side of the anomaly equation (5) is always a total derivative,

$$\epsilon^{\alpha_1\beta_1\cdots\alpha_n\beta_n}\,\mathrm{tr}\Big(F_{\alpha_1\beta_1}\cdots F_{\alpha_n\beta_n}\Big) \;=\; \partial_\mu\Omega^\mu_{(2n-1)} \tag{6}$$

where $\Omega^\mu_{(2n-1)}$ is some polynomial in gauge fields A^ν and $F^{\rho\sigma}$, for example

$$\begin{aligned}
\text{in } d=2,\ \Omega^\mu_{(1)} &= 2\epsilon^{\mu\nu}\,\mathrm{tr}(A_\nu) \quad [\text{abelian } A_\nu \text{ only}],\\
\text{in } d=4,\ \Omega^\mu_{(3)} &= 2\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(A_\nu F_{\rho\sigma} - \frac{2ig}{3}A_\nu A_\rho A_\sigma\right),\\
\text{in } d=6,\ \Omega^\mu_{(5)} &= 2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\left(A_\nu F_{\rho\sigma}F_{\alpha\beta} - igA_\nu A_\rho A_\sigma F_{\alpha\beta} - \frac{2g^2}{5}A_\nu A_\rho A_\sigma A_\alpha A_\beta\right),
\end{aligned} \tag{7}$$

etc., etc. The $\Omega^\mu_{(2n-1)}$ vectors are equivalent to $(2n-1)$–index totally antisymmetric tensors called the *Chern–Simons forms*, and those forms play many important roles in gauge theory and string theory. In particular, we may use the $\Omega^{\mu}_{(2n-1)}$ to define a conserved axial current

$$J^{\mu}_{A} \;\to\; J^{\mu}_{AC} \;=\; \overline{\Psi}\Gamma\gamma^{\mu}\Psi \;+\; \frac{1}{n!}\left(\frac{g}{4\pi}\right)^{n} \times \Omega^{\mu}_{(2n-1)}\,. \tag{8}$$

(Its conservation follows from eqs. (5) and (6).) However, the price of this current conservation is the loss of gauge invariance: the original axial current J^{μ}_{A} is gauge invariant, but the J^{μ}_{AC} is not.

You task is to verify eqs. (6) for $d = 2, 4, 6$.