# Renormalization Scheme Dependence

The running couplings such as $\lambda(E)$ depend not only on the energy scale $E$, but also on the specific rules we use to fix the finite parts of the $\delta^\lambda(E)$ and other counterterms. If we change those rules — collectively known as the renormalization scheme — then for the same energy scale $E$ we would get a slightly different running coupling $\lambda'(E) \neq \lambda(E)$. The difference is due to quantum corrections which usually start at one loop, thus

$$\lambda'(E) \;=\; \lambda(E) \;+\; O(\lambda^2(E)). \tag{1}$$

Corollary to this scheme-dependence of the running coupling $\lambda(E)$, the beta-function

$$\beta(\lambda) \;\equiv\; \frac{d\lambda(E)}{d\log E} \tag{2}$$

also depends on the renormalization scheme, $\beta'(\lambda') \neq \beta(\lambda)$. However, this dependence starts at the three-loop level; the one-loop and two-loop terms in the beta-function are the same in all renormalization schemes!

Before I prove this statement, let me make it precise. In a theory with a single running coupling $\lambda(E)$ (or $\alpha(E) = e^2(E)/4\pi$, or $g^2/4\pi$, or whatever), the beta-function is a power series

$$\beta(\lambda) \;=\; b_1\lambda^2 \;+\; b_2\lambda^3 \;+\; b_3\lambda^4 \;+\; \cdots \tag{3}$$

with some constant coefficients $b_1, b_2, b_3, \ldots$; each $b_\ell$ arises at the $\ell$–loop order of the perturbation theory. Now let's change the renormalization scheme (for the same theory) so the coupling becomes $\lambda'(E)$ and the beta-function becomes

$$\beta'(\lambda') \;=\; b'_1\lambda'^2 \;+\; b'_2\lambda'^3 \;+\; b'_3\lambda'^4 \;+\; \cdots \tag{4}$$

— a power series similar to (3), but maybe with different coefficients $b'_1, b'_2, b'_3, \ldots$.

**Theorem:** *the one-loop and two-loop coefficients are the same in all renormalization schemes,* $b'_1 = b_1$ *and* $b'_2 = b_2$*, but the three-loop and higher-loop coefficients are scheme dependent,* $b'_\ell \neq b_\ell$ *for* $\ell \geq 3$.

**Proof:** Let's spell out the relation (1) between the couplings $\lambda$ and $\lambda'$ as a power series

$$\lambda'(E) \;=\; \lambda(E) \;+\; C_1 \times \lambda^2(E) \;+\; C_2 \times \lambda^3 \;+\; \cdots \tag{5}$$

with some constant coefficients $C_1, C_2, \ldots$. Now consider the inverse couplings $1/\lambda(E)$ and

$$\frac{1}{\lambda'(E)} \;=\; \frac{1}{\lambda(E)} \;-\; C_1 \;+\; (C_1^2 - C_2) \times \lambda(E) \;+\; (-C_1^3 + 2C_1C_2 - C_3) \times \lambda^2(E) \;+\; \cdots. \tag{6}$$

These inverse couplings depend on energy according to

$$\begin{aligned} \frac{d}{\log E}\frac{1}{\lambda(E)} \;&=\; \frac{-1}{\lambda^2(E)} \times \left(\frac{d\lambda(E)}{d\log E} \;=\; \beta(\lambda(E))\right) \\ &=\; -b_1 \;-\; b_2 \times \lambda(E) \;-\; b_3 \times \lambda^2(E) \;-\; \cdots, \end{aligned} \tag{7}$$

and similarly

$$\frac{d}{\log E}\frac{1}{\lambda'(E)} \;=\; -b_1' \;-\; b_2' \times \lambda'(E) \;-\; b_3' \times \lambda'^2(E) \;-\; \cdots. \tag{8}$$

On the other hand, differentiating both sides of eq. (6) gives us

$$\begin{aligned} \frac{d}{\log E}\frac{1}{\lambda'(E)} \;&=\; \frac{d(1/\lambda')}{d\lambda} \times \left(\frac{d\lambda}{d\log E} \;=\; \beta(\lambda)\right) \\ &=\; \left(\begin{aligned} &\frac{-1}{\lambda^2(E)} \;-\; C_1 \times \frac{0}{\lambda(E)} \;+\; (C_1^2 - C_2) \times 1 \\ &\qquad + (-C_1^3 + 2C_1C_2 - C_3) \times 2\lambda(E) \;+\; \cdots \end{aligned}\right) \times \\ &\qquad \times \left(b_1 \times \lambda^2(E) \;+\; b_2 \times \lambda^3(E) \;+\; b_3 \times \lambda^4(E) \;+\; \cdots\right) \\ &=\; -b_1 \;-\; b_2 \times \lambda(E) \;-\; \left[b_3 - b_1(C_1^2 - C_2)\right] \times \lambda^2(E) \;-\; \cdots \\ &=\; -b_1 \;-\; b_2 \times \lambda'(E) \;-\; \left[b_3 - b_2C_1 - b_1(C_1^2 - C_2)\right] \times \lambda'^2(E) \;-\; \cdots. \end{aligned} \tag{9}$$

Comparing this formula to eq. (8) we immediately see that $b_1' = b_1$ and $b_2' = b_2$ but $b_3' = b_3 - b_2C_1 - b_1(C_1^2 - C_2) \neq b_3$, and it's obvious that the higher-order coefficients are also renormalization scheme dependent, $b_4' \neq b_4$, *etc. Quod erat demonstrandum.*

A similar theorem applies to theories with multiple couplings $g_i(E)$: Write all beta-functions $\beta_i$ as power series in $g_1(E),\ldots,g_n(E)$ with some numerical coefficients; the coefficients of all terms which arise at the one-loop or two-loop orders do not depend on the choice of the renormalization scheme, but the coefficients of the three-loop and higher-order terms are scheme-dependent.

## Minimal Subtraction

Over the years, field theorists invented all kinds of renormalization schemes. But since 1970's, the most popular schemes are the *Minimal Subtraction* (MS) and its close cousins $\overline{\mathrm{MS}}$, DR, and $\overline{\mathrm{DR}}$. Here are the rules for the MS scheme:

1. Use dimensional regularization to control the UV divergences.

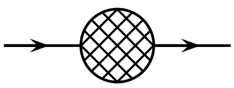
   Note: this rule is peculiar to the Minimal Subtraction and similar schemes. The other renormalization schemes do not care what the UV regulator is, you can use whatever regulator you like as long as it works (*i.e.*, regulates all the UV divergences and does not break symmetries that lead to Ward identities).

2. Identify the $\mu$ parameter of dimensional regularization

$$\frac{d^4p}{(2\pi)^4}\ \to\ \mu^{4-D}\times\frac{d^Dp}{(2\pi)^D} \tag{10}$$

   with the energy scale $E$ of the renormalization group. This identification sets the ubiquitous logarithms $\log(\mu^2/E^2)$ to zero.

3. In general, the overall UV divergence of some $L$-loop amplitude is a degree-$L$ polynomial in $1/\epsilon$, for example



$$\supset \not{p}\times g^{2L}\times\left(\frac{A_L}{\epsilon^L}+\frac{A_{L-1}}{\epsilon^{L-1}}+\cdots+\frac{A_1}{\epsilon}+\mathit{a\_finite\_function\_of}(p^2)\right) \tag{11}$$

   for some constants $A_L,\ldots,A_1$, and to cancel such a divergence we need an $L$-loop-order counterterm

$$\delta^Z_{L\,\mathrm{loops}}\ =\ g^{2L}\times\left(\frac{A_L}{\epsilon^L}+\frac{A_{L-1}}{\epsilon^{L-1}}+\cdots+\frac{A_1}{\epsilon}+A_0\right). \tag{12}$$

   In this counterterm, the coefficients $A_L,\ldots,A_1$ are completely determined by the UV divergence of the $L$-loop diagrams, but the finite free term $A_0$ is not: its value follows not from the divergence but from the renormalization scheme we use for the amplitude (11).

In the MS scheme, we do not impose any conditions on amplitudes. Instead, we simply set $A_0 = 0$. Likewise, the finite parts of all the other counterterms are set to zero.

This is called the *Minimal Subtraction* because all the counterterms do is to subtract the pole at $\epsilon = 0$; the finite part of a divergent amplitude is whatever the loop diagrams produce, the counterterms do not mess with it.

$\star \quad \star \quad \star$

In general, an $L$-loop counterterm in the MS renormalization scheme comprises poles in $1/\epsilon$ of orders 1 through $L$. However, there are recursion relations for all the higher poles $1/\epsilon^2$, $1/\epsilon^3$, *etc.*, in terms of the lower-degree poles of the lower-loop-order counterterms. For example, the $1/\epsilon^2$ pole of a 2-loop counterterm can be obtained from the $1/\epsilon$ poles of the 1-loop counterterms without doing any 2-loop calculations. Only the simple $1/\epsilon$ poles have to be calculated the hard way for each loop order: QFT is hard, but not quite as hard as it could be.

In this section of the notes, I'll write the recursion relations for the $1/\epsilon^2$, *etc.*, poles for the counterterms combinations used for calculating the beta-functions. I'll also write down a closed formula for the beta-function(s) in terms of the simple $1/\epsilon$ poles only. Recursion relations for the other counterterms are left out as an optional exercise for the students. (In case you have nothing else to do during the summer break, or if you need them for your own research.)

Let's start with the $\lambda\phi^4$ theory. As explained in class, the renormalized coupling $\lambda(\mu)$ is related to the bare coupling $\lambda_b$ according to

$$\lambda_b \;=\; \frac{\lambda(\mu) + \delta^\lambda(\mu)}{(1 + \delta^Z(\mu))^2} \tag{13}$$

In the MS scheme, the counterterms are given by power series

$$\begin{aligned} \delta^\lambda &= \sum_{L=1}^{\infty} \lambda^{L+1} \times \sum_{k=1}^{L} \frac{A_{L,k}}{\epsilon^k}\,, \\ \delta^Z &= \sum_{L=1}^{\infty} \lambda^{L} \times \sum_{k=1}^{L} \frac{B_{L,k}}{\epsilon^k} \end{aligned} \tag{14}$$

with some constant coefficients $A_{L,k}$ and $B_{L,k}$. In the perturbative series like (14) we should treat the coupling $\lambda$ as infinitesimally small, so small that even $\lambda/\epsilon$ is small despite the eventual $\epsilon \to 0$ limit. In other words, we should take the $\lambda \to 0$ limit *before* taking $\epsilon$ to zero. In this limit, the right hand side of eq. (13) also becomes a power series

$$\frac{\lambda(\mu) + \delta^\lambda(\mu)}{(1+\delta^Z(\mu))^2} = \lambda(\mu) + \sum_{L=1}^{\infty} \lambda^{L+1}(\mu) \times \sum_{k=1}^{L} \frac{C_{L,k}}{\epsilon^k} \tag{15}$$

where the coefficients $C_{L,k}$ are given by polynomials in $A_{L',k'}$ and $B_{L',k'}$ with $L' \leq L$ and $k' \leq k$,

$$C_{1,1} = A_{1,1} - 2B_{1,1}\,, \qquad C_{2,1} = A_{2,1} - 2B_{1,1}\,, \qquad C_{2,2} = A_{22} - A_{11}B_{11} + 3B_{11}^2 - 2B_{2,2}\,, \ldots. \tag{16}$$

Let's re-organize the series (15) by summing over the loop order $L$ before summing over the pole degree $k$, thus

$$\frac{\lambda(\mu) + \delta^\lambda(\mu)}{(1+\delta^Z(\mu))^2} = \lambda(\mu) + \sum_{k=1}^{\infty} \frac{f_k(\lambda(\mu))}{\epsilon^k} \tag{17}$$

where

$$f_k(\lambda) \equiv \sum_{L=k}^{\infty} C_{L,k} \lambda^{L+1}. \tag{18}$$

In a moment, we shall see that all the higher-pole coefficients $f_2(\lambda)$, $f_3(\lambda)$, *etc.*, are completely determined by the simple-pole coefficient $f_1(\lambda)$, and there is a simple formula for the beta-function $\beta(\lambda)$ in terms of just the $f_1(\lambda)$. But before I prove this statement, let me mention that for $k = 1$, the $C_{L,1}$ is simply $A_{L,1} - 2B_{L,1}$, hence

$$f_1(\lambda) = \text{residue of the simple } \frac{1}{\epsilon} \text{ pole of } \delta^\lambda - 2\lambda\delta^Z, \tag{19}$$

and that's all we shall need to calculate the beta-function $\beta(\lambda)$.

Eq. (17) spells out the right hand side of eq. (13) for any dimension $D = 4 - 2\epsilon \neq 4$. Now let's take a closer look at the left hand side for $D \neq 4$. The problem with the bare coupling $\lambda_b$ is that it's dimensionless only in $D = 4$ dimensions, but in other $D$ it has dimensionality mass$^\Delta$ where

$$\Delta \;=\; D \;-\; 4\dim[\Phi] \;=\; D \;-\; 4\frac{D-2}{2} \;=\; 4 - D \;=\; 2\epsilon. \tag{20}$$

The running coupling $\lambda(\mu)$ suffers from a similar problem, but we can make it dimensionless for any $D \neq 4$ by rescaling $\lambda(\mu) \to [\lambda(\mu)]^{\text{dimensionless}} \times \mu^{2\epsilon}$. In fact, such rescaling happens automatically when we include the $\mu^{2\epsilon}$ factors in the dimensionally regularized momentum integrals (10), so in eq. (17) $\lambda(\mu)$ is already dimensionless.

But when we apply a similar rescaling to the bare coupling $\lambda_b$, we make the coupling dimensionless but $\mu$-dependent. In class, we have derived the beta-function from the fact that $\lambda_b$ was divergent but $E$-independent, but now that we work in $D \neq 4$ dimensions, we should use

$$\lambda_b(\mu) \;=\; \frac{\text{divergent constant}}{\mu^{2\epsilon}} \tag{21}$$

on the left hand side of eq. (13). The right hand side of eq. (13) is spelled out in eq. (17); combining all these formulae together, we arrive at

$$\frac{\text{divergent constant}}{\mu^{2\epsilon}} \;=\; \lambda_b \;=\; \frac{\lambda(\mu) + \delta^\lambda(\mu)}{(1+\delta^Z(\mu))^2} \;=\; \lambda(\mu) \;+\; \sum_{k=1}^{\infty} \frac{f_k(\lambda(\mu))}{\epsilon^k}\,. \tag{22}$$

Now let's differentiate both sides of eq. (22) with respect to $\log\mu$. The right hand side depends on $\mu$ only via $\lambda(\mu)$, hence

$$\frac{d}{d\log\mu}\left(\lambda(\mu) + \sum_{k=1}^{\infty} \frac{f_k(\lambda(\mu))}{\epsilon^k}\right) \;=\; \frac{d\lambda}{d\log\mu} \times \frac{d}{d\lambda}\Big(\cdots\Big) \;=\; \beta(\lambda(\mu)) \times \left(1 + \sum_{k=1}^{\infty} \frac{f_k'(\lambda(\mu))}{\epsilon^k}\right) \tag{23}$$

where $f_k'(\lambda)$ is $df_k/d\lambda$. On the left hand side of eq. (22), the $\mu$-dependence is explicit but we don't know the constant coefficient. Instead, we may obtain it from the eq. (22) itself, thus

$$\frac{d}{d\log\mu}\left(\frac{\text{const}}{\mu^{2\epsilon}}\right) \;=\; -2\epsilon \times \frac{\text{same const}}{\mu^{2\epsilon}} \;=\; -2\epsilon \times \left(\lambda(\mu) + \sum_{k=1}^{\infty} \frac{f_k(\lambda(\mu))}{\epsilon^k}\right) \tag{24}$$

and therefore

$$-2\epsilon\lambda \;-\; 2\sum_{k=1}^{\infty}\frac{f_k(\lambda)}{\epsilon^{k-1}} \;=\; \beta(\lambda)\times\left(1+\sum_{k=1}^{\infty}\frac{f'_k(\lambda)}{\epsilon^k}\right). \tag{25}$$

At this point, let's treat both sides of eq. (25) as Laurent power series[⋆] in $\epsilon$. On the right hand side, the beta-function $\beta(\lambda)$ depends on the spacetime dimension, so we should treat it as $\beta(\lambda,\epsilon)$ and expand

$$\beta(\lambda,\epsilon) \;=\; \beta_0(\lambda) \;+\; \epsilon\times\beta_1(\lambda) \;+\; \epsilon^2\times\beta_2(\lambda) \;+\; \cdots. \tag{26}$$

Note that only non-negative powers of $\epsilon$ appear in this expansion because the beta-function does not have a singularity at $D=4$. Thus, eq. (22) becomes

$$-2\epsilon\lambda \;-\; 2\sum_{k=1}^{\infty}\frac{f_k(\lambda)}{\epsilon^{k-1}} \;=\; \left(\sum_{n=0}^{\infty}\beta_n(\lambda)\times\epsilon^{+n}\right)\times\left(1+\sum_{k=1}^{\infty}\frac{f'_k(\lambda)}{\epsilon^k}\right), \tag{27}$$

and the coefficients of similar powers of $\epsilon$ should be equal on both sides of this equation. In particular, since the left hand side does not contain any powers of $\epsilon$ greater than $+1$, the right hand side should not contain them either, and this can happen only if the expansion (26) for the beta-functions stops after the linear term,

$$\beta(\lambda,\epsilon) \;=\; \beta_0(\lambda) \;+\; \epsilon\times\beta_1(\lambda) \;+\; \text{nothing else}. \tag{28}$$

This fact greatly simplifies eq. (27) — it becomes

$$-2\epsilon\lambda \;-\; 2f_1 \;-\; 2\sum_{k=2}^{\infty}\frac{f_k}{\epsilon^{k-1}} \;=\; \epsilon\beta_1 \;+\; \beta_0 \;+\; \beta_1\times f'_1 \;+\; \sum_{k=2}^{\infty}\frac{\beta_1 f'_k}{\epsilon^{k-1}} \;+\; \sum_{k=1}^{\infty}\frac{\beta_0 f'_k}{\epsilon^k}, \tag{29}$$

and now it's easy to compare similar powers of $\epsilon$ on both sides. Starting with $\epsilon^{+1}$ and going down, we have

$$\beta_1(\lambda) \;=\; -2\lambda, \quad \text{exactly,} \tag{30}$$

⋆ Unlike the Taylor series which sums up only non-negative powers of some variable, the Laurent series includes both positive and negative powers. A function $f(z)$ that's singular at $z=0$ but is analytic in some ring $r_1<|z|<r_2$ in complex $z$ plane can be expanded into a Laurent series in both positive and negative powers of $z$.

$$\begin{aligned}
\beta_0(\lambda) \;&=\; -2f_1(\lambda) \;-\; \beta_1(\lambda)\times f_1'(\lambda), && (31)\\
-\beta_1(\lambda)\times f_2'(\lambda) \;-\; 2f_2(\lambda) \;&=\; \beta_0(\lambda)\times f_1(\lambda), && (32)\\
\cdots\cdots\cdots\cdots\cdots\cdots\; &\;\cdots\cdots\cdots\cdots\cdots\cdots\\
-\beta_1(\lambda)\times f_k'(\lambda) \;-\; 2f_k(\lambda) \;&=\; \beta_0(\lambda)\times f_{k-1}(\lambda). && (33)
\end{aligned}$$

These formulae give us everything we want to know in terms of the $f_1(\lambda)$ function, which summarizes the simple poles in the $\delta^\lambda$ and $\delta^Z$ counterterms according to eq. (19). In particular, eqs. (30) and (31) give us the beta-function in any spacetime dimension $D = 4 - 2\epsilon$,

$$\beta(\lambda) \;=\; (D-4)\times\lambda \;+\; \left(2\lambda\frac{d}{d\lambda} - 2\right) f_1(\lambda). \tag{34}$$

Thus, if in the MS regularization scheme

$$\delta^\lambda \;-\; 2\lambda\delta^Z \;=\; \frac{c_1\lambda^2 + c_2\lambda^3 + c_3\lambda^4 + \cdots}{\epsilon} \;+\; \text{higher poles}, \tag{35}$$

then

$$\beta(\lambda) \;=\; (D-4)\times\lambda \;+\; 2c_1\lambda^2 \;+\; 4c_2\lambda^3 \;+\; 6c_3\lambda^4 \;+\; \cdots. \tag{36}$$

As to eqs. (32), *etc.*, they gives us the recursion relations for the $f_k(\lambda)$ functions for the higher $1/\epsilon^k$ poles with $k \geq 2$. Specifically,

$$\left(2\lambda\frac{d}{d\lambda} - 2\right) f_k(\lambda) \;=\; \beta_0(\lambda)\times\frac{df_{k-1}(\lambda)}{d\lambda}\,. \tag{37}$$

These differential equations completely determine the $f_k(\lambda)$ functions once we impose the initial conditions $f_k = O(\lambda^{k+1})$ for $\lambda \to 0$.

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Now consider a generic QFT with several couplings $g_s(\mu)$, $s = 1, \ldots, n$. Similar to $\lambda(\mu)$, we make all $g_s(\mu)$ dimensionless by multiplying them by appropriate powers of $\mu$. Then for each coupling we have an equation similar to eq. (22):

$$g_{s,\text{bare}} \;=\; \frac{\text{const}}{\mu^{\Delta_s}} \;=\; \frac{g_s(\mu)+\delta_s^g(\mu)}{\prod\limits_{\substack{\text{appropriate}\\ \text{fields } i}} (1+\delta_i^Z(\mu))^{1/2}} \;=\; g_s(\mu) \,+\sum_{k=1}^{\infty}\frac{1}{\epsilon^k}\,f_s^{(k)}(g_1(\mu),\ldots,g_n(\mu)). \tag{38}$$

On the right hand side here

$$f_s^{(1)}(g_1,\ldots,g_n) \;=\; \text{coefficient of the simple } \frac{1}{\epsilon} \text{ pole of } \left[\delta_s^g \,-\, \frac{g_s}{2}\times \sum_{\substack{\text{appropriate}\\ \text{fields } i}} \delta_i^Z\right] \tag{39}$$

while the other $f_s^{(k>1)}$ follow from the higher-order poles in the counterterms. The specific formulae for the $f_s^{(2)}$, *etc.*, in terms of those higher poles are rather complicated, but fortunately we do not need them to calculate the beta functions.

On the left hand side of eq. (38), $\Delta_s$ is the canonical dimensionality of the bare coupling $g_s$ in $D = 4 - 2\epsilon$ dimensions. In general, different couplings have different dimensionalities, but fortunately they are always linear functions of spacetime dimension $D$ and hence of the $\epsilon$,

$$\Delta_s(\epsilon) \;=\; \Delta_s^{(0)} \;+\; K_s\times\epsilon, \quad \text{exactly.} \tag{40}$$

For the renormalizable couplings $\Delta_s^{(0)} = 0$ while $K_s = \text{valence(vertex)} - 2$: for the gauge and Yukawa couplings $K_s = 3 - 2 = 1$ while for the 4–scalar coupling $K_s = 2$.

Similar to eq. (28), linearity of the $\Delta_s$ dependence on the $\epsilon$ makes all the beta-functions $\beta(g_a,\ldots,g_n;\epsilon)$ exactly linear with respect to epsilon, which helps us to calculate them in terms of the $f_s^{(1)}$ functions. Indeed, taking the derivative of both sides of eq. (38) with respect to the $\log\mu$, we obtain

$$\begin{aligned} -(\Delta_s^{(0)} \,+\, \epsilon K_s)\times\left(g_s \,+\sum_{k=1}^{\infty}\frac{1}{\epsilon^k}\times f_s^{(k)}(g_1,\ldots,g_n)\right) \;&=\; \\ =\sum_{p=1}^{n}\beta_p(g_1,\ldots,g_n;\epsilon)\times&\left[\delta_{p,s} \,+\sum_{k=1}^{\infty}\frac{1}{\epsilon^k}\times\frac{\partial f_s^{(k)}(g_1,\ldots,g_n)}{\partial g_p}\right]. \end{aligned} \tag{41}$$

And now we treat both sides as Laurent series in powers of $\epsilon$ and compare coefficients of similar powers on both sides. Since the left hand side does not include any powers greater than $\epsilon^{+1}$, the right hand side should not have them either, thus

$$\beta_s(g_1,\ldots,g_n;\epsilon) \;=\; \beta_s^{(0)}(g_1,\ldots,g_n) \;+\; \epsilon\times\beta_s^{(1)}(g_1,\ldots,g_n) \;+\; \text{nothing else}, \tag{42}$$

Consequently, matching powers of $\epsilon^{+1},\epsilon^0,\epsilon^{-1},\ldots$, we obtain

$$\begin{aligned}
-K_s\times g_s \;&=\; \beta^{(1)_s}\,, && (43)\\
-\Delta_s^{(0)}\times g_s \;-\; K_s\times f_s^{(1)} \;&=\; \beta_s^{(0)} \;+\; \sum_p \beta_p^{(1)}\times\frac{\partial f_s^{(1)}}{\partial g_p}\,, && (44)\\
-\Delta_s^{(0)}\times f_s^{(1)} \;-\; K_s\times f_s^{(2)} \;&=\; \sum_p\left[\beta_p^{(1)}\times\frac{\partial f_s^{(2)}}{\partial g_p} \;+\; \beta_p^{(0)}\times\frac{\partial f_s^{(1)}}{\partial g_p}\right], && (45)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
-\Delta_s^{(0)}\times f_s^{(k)} \;-\; K_s\times f_s^{(k+1)} \;&=\; \sum_p\left[\beta_p^{(1)}\times\frac{\partial f_s^{(k+1)}}{\partial g_p} \;+\; \beta_p^{(0)}\times\frac{\partial f_s^{(k)}}{\partial g_p}\right]. && (46)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

The first two equations here — (43) and (44) — give us the exact formulae for all the $\beta$ functions in terms of the

$$f_s^{(1)}(g_1,\ldots,g_n) \;=\; \text{coefficient of the simple } \frac{1}{\epsilon} \text{ pole of } \left[\delta_s^g \;-\; \frac{g_s}{2}\times\sum_{\substack{\text{appropriate}\\ \text{fields } i}}\delta_i^Z\right], \tag{39}$$

namely

$$\beta_s(g_1,\ldots,g_n;D) \;=\; -\Delta_s(D)\times g_s \;+\; \left(\sum_{p=1}^n K_p g_p\frac{\partial}{\partial g_p} \;-\; K_s\right) f_s^{(1)}(g_1,\ldots,g_n). \tag{47}$$

When all the couplings are renormalizable, this formula simplifies to

$$\beta_s(g_1,\ldots,g_n;D) \;=\; -\Delta_s(D)\times g_s \;+\; 2\hat{L}\,f_s^{(1)}(g_1,\ldots,g_n) \tag{48}$$

where $\hat{L}$ is the operator counting the number of loops giving rise to each term in the $f_s^{(1)}$. As promised, in the MS regularization scheme, the simple $1/\epsilon$ poles of the appropriate counterterms completely determine all the beta-functions of the theory. As to the coefficients $f_s^{(k>1)}$

of the higher-order poles $1/\epsilon^k$ in the same counterterms, they follow from the simple pole coefficients via the recursion relations (46), or in a more compact form,

$$\left(\sum_{p=1}^{n} K_p g_p \frac{\partial}{\partial g_p}\,-\,K_s\right) f_s^{(k+1)}(g_1,\ldots,g_n)\;=\;\left(\sum_p \beta_s^{(0)}\times\frac{\partial}{\partial g_p}\,-\,\Delta_s^0\right) f_s^{(k)}(g_1,\ldots,g_n). \quad (49)$$

When all the couplings are renormalizable, this formula becomes even simpler:

$$2\hat{L} f_s^{(k+1)}(g_1,\ldots,g_n)\;=\;\left(\sum_p \beta_s^{(0)}(g_1,\ldots,g_n)\times\frac{\partial}{\partial g_p}\right) f_s^{(k)}(g_1,\ldots,g_n). \quad (50)$$

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Instead of the original Minimal Subtraction renormalization scheme (MS), people often use the Modified Minimal Subtraction scheme ($\overline{\mathrm{MS}}$, pronounced MS-bar). In this scheme, the $L$-loop counterterms are degree-$L$ polynomials — without the free term — in

$$\frac{1}{\bar{\epsilon}}\;\stackrel{\mathrm{def}}{=}\;\frac{1}{\epsilon}\,-\,\gamma_E\,+\,\log(4\pi) \quad (51)$$

instead of $1/\epsilon$. This modification makes the regularized net amplitudes somewhat simpler because it subtracts the numerical constants that usually accompany the $1/\epsilon$ poles.

There are also DR and $\overline{\mathrm{DR}}$ regularization schemes which are often used in supersymmetric theories. These schemes work similarly to the MS and $\overline{\mathrm{MS}}$ but use a different 'flavor' of dimensional regularization called *dimensional reduction*: all momenta live in $D=4-2\epsilon$ dimensions, but the vector fields keep all 4 components. Physically, such a reduced 4D vector field comprises one species of a $D$-dimensional vector plus $2\epsilon$ species of scalars with the same mass and charge. Unlike the original 't Hooft's dimensional regularization, the dimensional reduction does not break the supersymmetry; apart from that, the difference is usually unimportant.